Welcome to MTH 211 !

Calculus III.

Calc I & II are based on functions

$$f : \mathbb{R} \to \mathbb{R}$$

one real input — one real output

Calc III is based on functions

$$f : \mathbb{R}^m \to \mathbb{R}^n$$

m real inputs — n real outputs

where $m, n = 1, 2$ or 3. This is more relevant to the real world, which is 3D.

One-varinble calulus was invented in the 1600s to study physics (specifically, gravity).

Multi-variable calculus was invented in the 1800s to study electricity & magnetism, which is described in terms of:

- gradient of a scalar field
- curl of a vector field
- divergence of a vector field

At the very end of the course we will discuss the meaning of "Maxwell's equations" for electro-magnetism:

$$\begin{cases} \nabla \cdot B = 0 \\ \nabla \cdot E = \rho \\ \nabla \times E = -\frac{\partial}{\partial t} B \\ \nabla \times B = \frac{\partial}{\partial t} E + J \end{cases}$$

Today "vector calculus" is crucial for many areas of science & computation.

This Week : Chapter 2.

But first a quick survey of Chap 1.

Let's consider a function

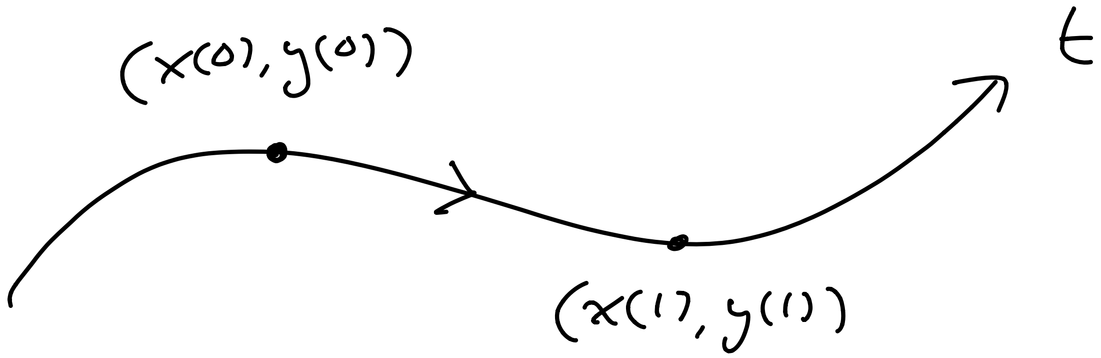
$$f : \mathbb{R} \to \mathbb{R}^2$$

one real input — two real outputs

Today we'll use the notation

$$f(t) = (x(t), y(t))$$

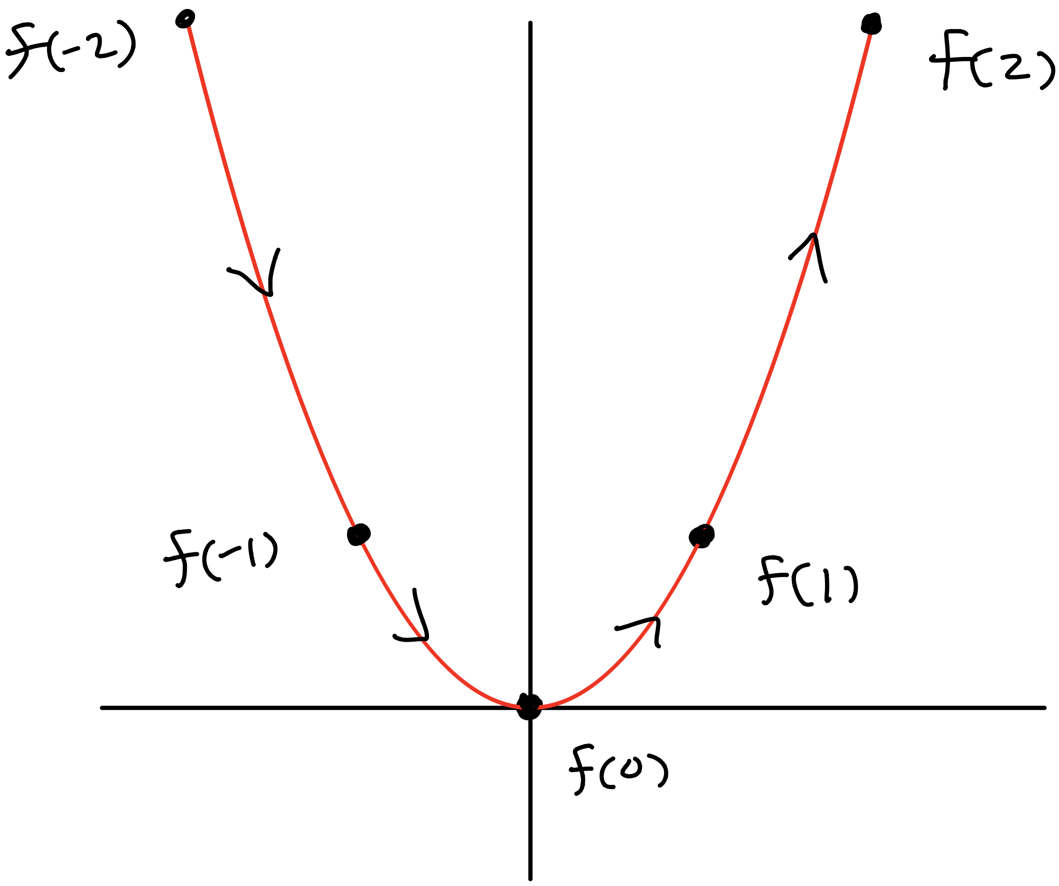
Intuition : t is "time" and $(x(t), y(t))$ is a <u>moving point</u> in the real x, y plane $\mathbb{R}^2$.



Examples :

- $f(t) = (x(t), y(t)) = (t, t^2)$

What does it look like ?



It's a parabola!

We can find the equation of the parabola by "eliminating t":

$$\begin{cases} x = t & (1) \\ y = t^2 & (2) \end{cases}$$

Square both sides of (1):

$$x^2 = t^2$$

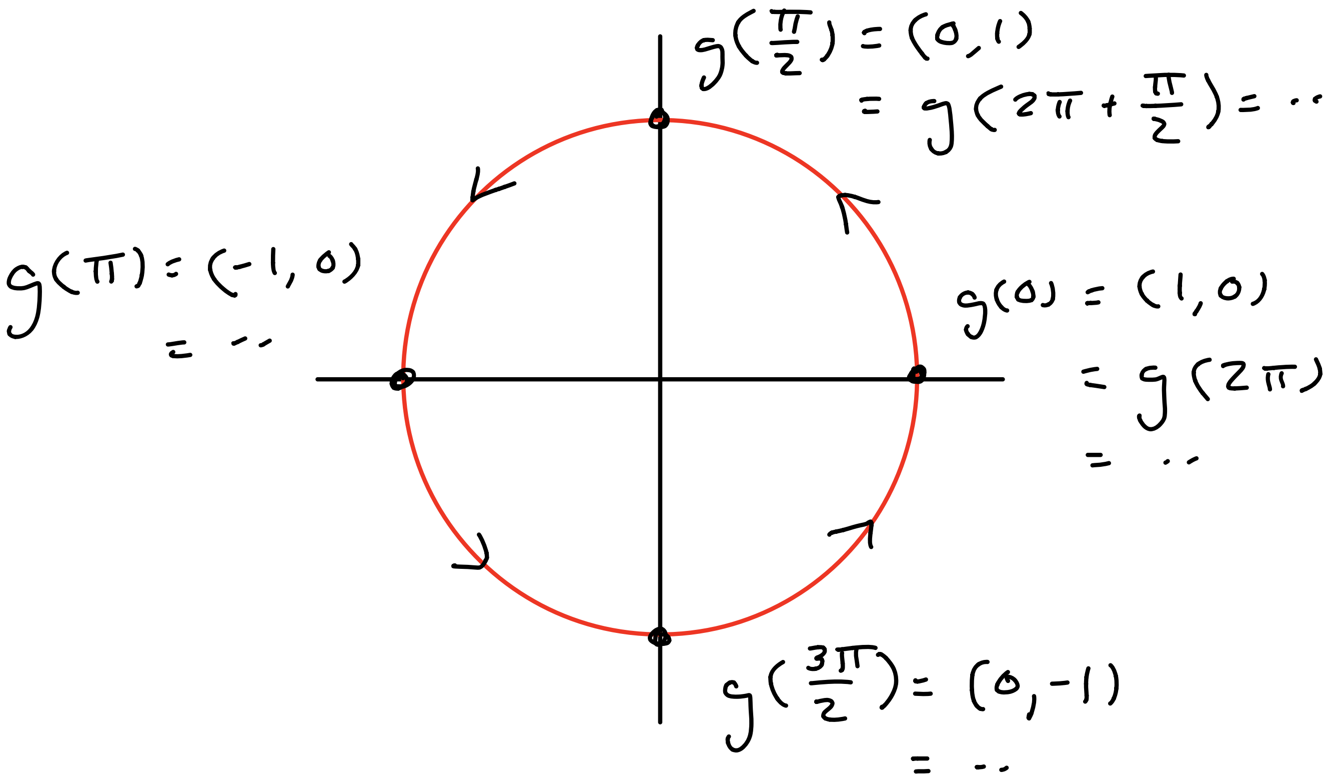
Substitute this into (2):

$$y = x^2.$$

- $g(t) = (x(t), y(t))$
  $= (\cos t, \sin t)$

What does it look like?

$g(t)$ travels counterclockwise around the unit circle:



We can find the equation of the circle by "eliminating t".

Use the Pythagorean theorem:

$$\cos^2 t + \sin^2 t = 1$$

$$x^2 + y^2 = 1.$$

Recall from Calc I: If $f(t)$ is the position of a particle at time t, then $f'(t) = \frac{df}{dt}$ is the "instantaneous velocity" of the particle at time t.

Definition: Given a function $f: \mathbb{R} \to \mathbb{R}^2$ written as

$$f(t) = (x(t), y(t)),$$

we define the derivative $f': \mathbb{R} \to \mathbb{R}^2$ by

$$f'(t) = (x'(t), y'(t))$$

$$\frac{dF}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right).$$

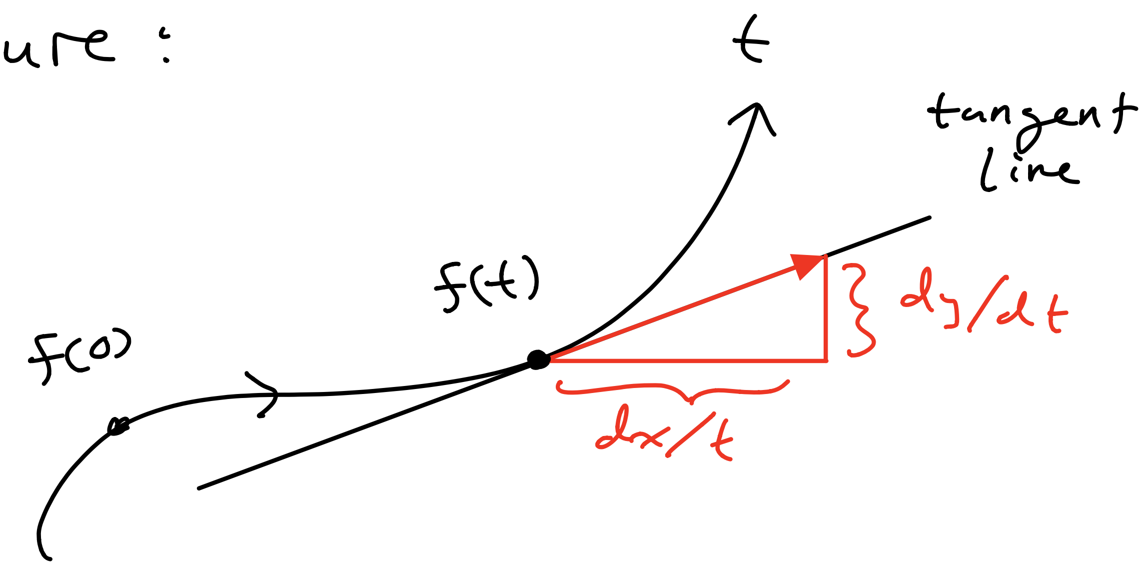
We call this the "instantaneous

velocity" of the parametrized path $f(t)$ at time t.

New Idea: The velocity of a path is a vector.

[Tomorrow: Chapter 2, Vectors.]

Picture:



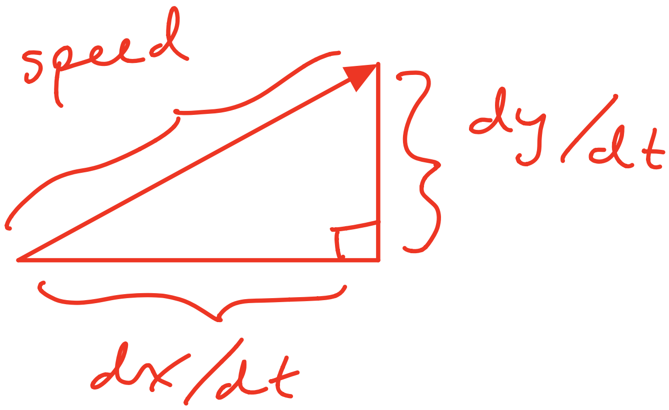
The red arrow is the velocity vector at time t. The slope of the tangent line is

$$\frac{\text{rise}}{\text{run}} = \frac{dy/dt}{dx/dt} = \frac{dy}{dx},$$

which looks correct!

The velocity is an arrow $f'(t)$.

The speed is the length of this arrow:



$$\text{speed}^2 = (dx/dt)^2 + (dy/dt)^2$$

$$\text{speed} = +\sqrt{(dx/dt)^2 + (dy/dt)^2}$$

$$= +\sqrt{(x'(t))^2 + (y'(t))^2}$$

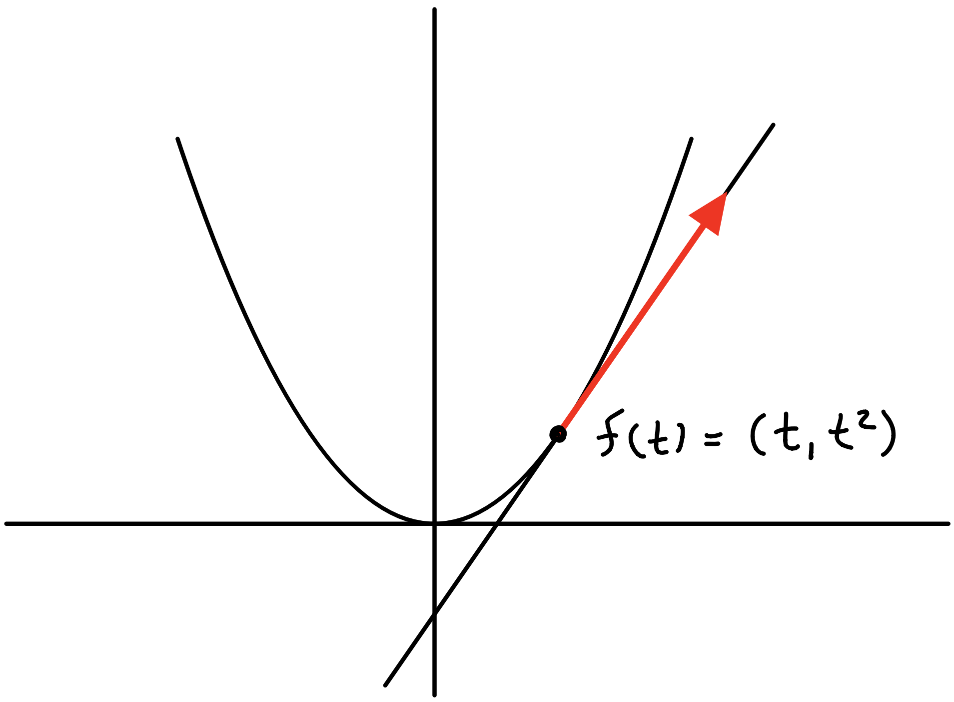
= the "instantaneous speed" of the particle at time t.

[velocity is a vector
speed is a number]

Examples:

- $f(t) = (x(t), y(t)) = (t, t^2)$

  $f'(t) = (x'(t), y'(t)) = (1, 2t)$



slope of the tangent at the point $f(t) = (t, t^2)$ is

$$\frac{\text{rise}}{\text{run}} = \frac{y'(t)}{x'(t)} = \frac{2t}{1} = 2t.$$

The <u>speed</u> at time t is

$$\text{speed} = \sqrt{(x'(t))^2 + (y'(t))^2}$$

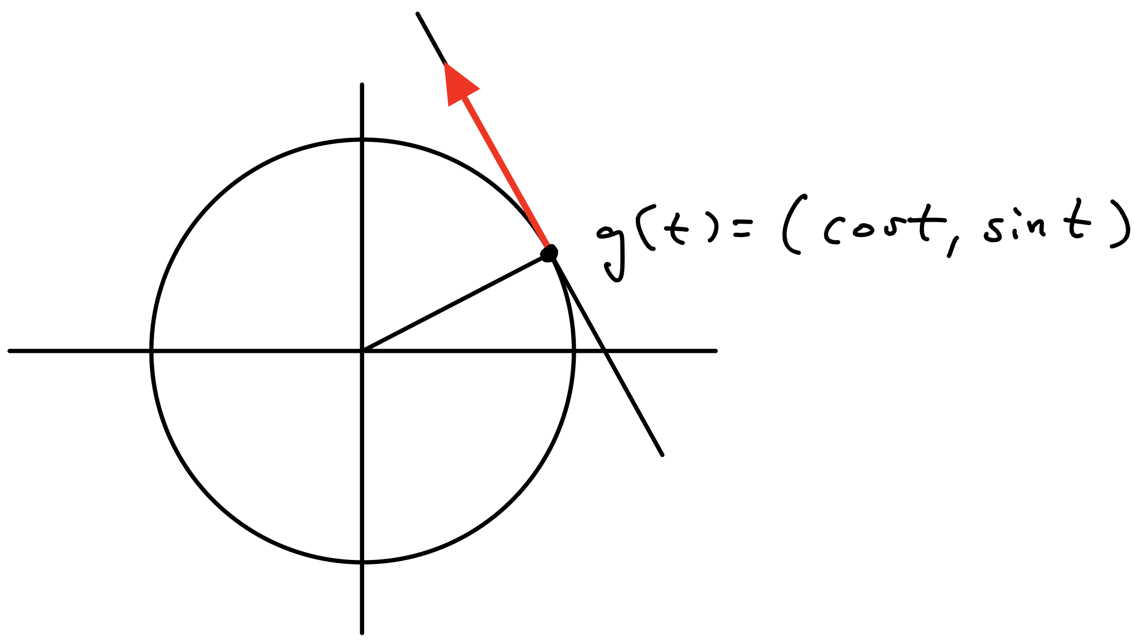
$$= \sqrt{1^2 + (2t)^2}$$

$$= \sqrt{1+4t^2}$$

So speed $= 1$ at time $t=0$, then it gets faster & faster.

- $g(t) = (x(t), y(t)) = (\cos t, \sin t)$

$g'(t) = (x'(t), y'(t)) = (-\sin t, \cos t)$.



velocity is always tangent to the circle. The speed is

$$\sqrt{(x'(t))^2 + (y'(t))^2}$$

$$= \sqrt{(-\sin t)^2 + (\cos t)^2}$$

$$= \sqrt{\sin^2 t + \cos^2 t}$$

$$= \sqrt{1}$$

$$= 1.$$

The particle moves counter-clockwise around the unit circle with "constant unit speed".

Last topic for today: Arc Length.

Recall from Calc I: If $s(t)$ is the speed of a particle at time t, then the distance traveled between $t=a$ & $t=b$ is

$$\text{distance} = \int_a^b \text{speed}\, dt$$

$$= \int_a^b s(t)\, dt$$

The same idea holds for parametrized paths in the plane:

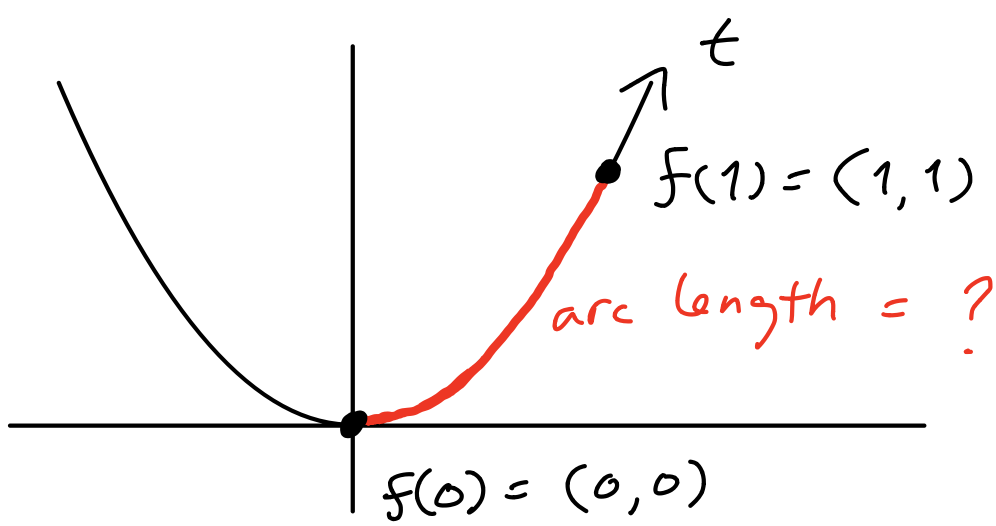
If a particle has position $f(t) = (x(t), y(t))$ at time t, then the distance traveled between times $t=a$ & $t=b$ is

$$\underset{\text{(arc length)}}{\text{distance}} = \int_a^b \text{speed}\, dt$$

$$= \int_a^b \sqrt{(x'(t))^2 + (y'(t))}\, dt$$

Examples :

- Find the arc length of the parabola $f(t) = (t, t^2)$ between times $t=0$ & $t=1$.



Answer:

$$\text{arc length} = \int_0^1 \text{speed}\, dt$$

$$= \int_0^1 \sqrt{1+4t^2}\, dt$$

Do you know how to compute this integral? Neither do I.

Computer:

$$\int_0^1 \sqrt{1+4t^2}\, dt \approx 1.479\ldots$$

[Moral: Integrals in arc length computations are usually very tricky ...]

- Compute the circumference of the unit circle.

Use parametrization

$$g(t) = (\cos t, \sin t)$$

with constant speed 1. The circumference is the arc length between times $t = 0$ & $t = 2\pi$:

$$\text{circumference} = \int_0^{2\pi} \text{speed}\, dt$$

$$= \int_0^{2\pi} 1\, dt = 2\pi,$$

as expected!