HW3 : up today, due Mon.

Quiz 3 : Tues.

Chapter 4 : Theory of Gradients.

Today : Linear Approximation & Multivariable Chain Rule.

Recall from Calc I & II :

- The Chain Rule.

Given $f, g : \mathbb{R} \to \mathbb{R}$ we have a composite function $f \circ g : \mathbb{R} \to \mathbb{R}$ defined by

$$(f \circ g)(x) = f(g(x))$$

The derivative is given by the so-called "chain rule":

$$\frac{d}{dx} f(g(x)) = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}$$

More abstractly:

$$[f \circ g]' = (f' \circ g) \cdot g'$$

- Linear Approximation.

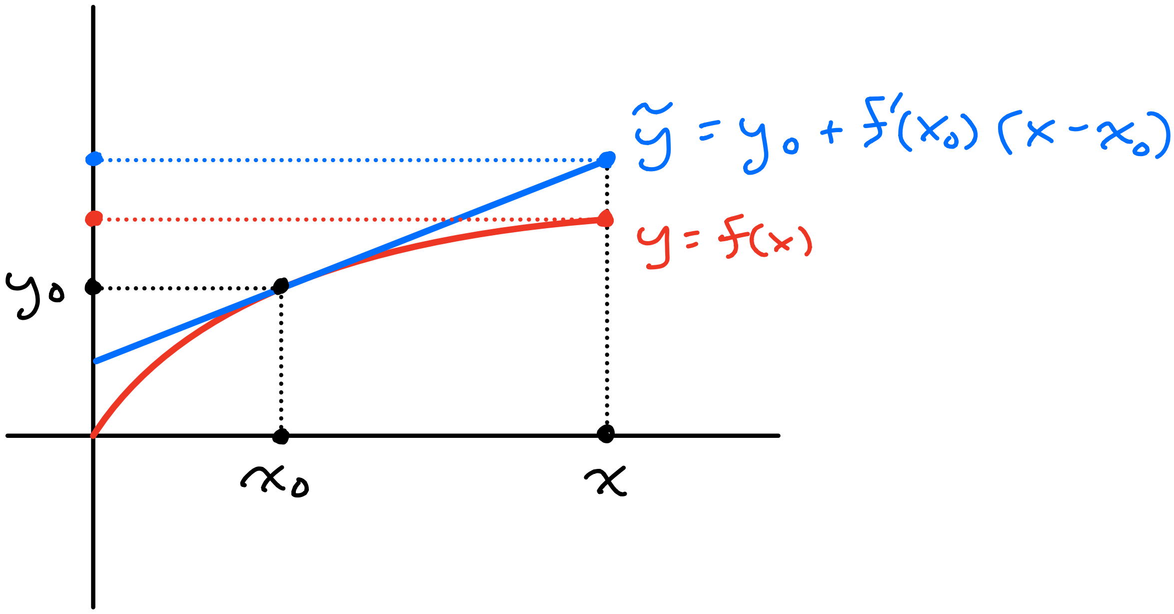
Given $f : \mathbb{R} \to \mathbb{R}$ & number $x_0 \in \mathbb{R}$, we have a Taylor series expansion:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \frac{1}{6} f'''(x_0)(x - x_0)^3 + \cdots$$

We obtain approximations to $f(x)$ by stopping the series early. The "best linear approximation" is

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

This approximation is good when $x \approx x_0$. The meaning of this has to do with the tangent line:



we can view

$$\tilde{y} = f(x_0) + f'(x_0)(x - x_0)$$
$$= y_0 + f'(x_0)(x - x_0)$$

as an approximation to the true value of y:

$$y = f(x)$$

we can also express this as

$$y - y_0 \approx f'(x_0)(x - x_0)$$
$$\Delta y \approx f'(x_0)\, \Delta x$$

change in y change in x

If the changes are very small, we can express this as an equation of "differential forms"

$$dy = f'(x_0)\, dx$$

$$dy = \frac{dy}{dx} \cdot dx$$

That looks correct!

This is just another way to state the chain rule.

Let's try to generalize this idea.

- Let $\vec{r} : \mathbb{R} \to \mathbb{R}^2$ & $f : \mathbb{R} \to \mathbb{R}$.

Then we get a composite function $\vec{r} \circ f : \mathbb{R} \to \mathbb{R}^2$ defined by

$$(\vec{r} \circ f)(t) = \vec{r}(f(t))$$

$$= \langle x(f(t)), y(f(t)) \rangle$$

And the derivative is

$$
\begin{aligned}
(\vec{r} \circ f)'(t) &= \langle (x \circ f)'(t), (y \circ f)'(t) \rangle \\
&= \langle x'(f(t))\, f'(t), y'(f(t))\, f'(t) \rangle \\
&= \langle x'(f(t)), y'(f(t)) \rangle\, f'(t) \\
&= \vec{r}\,'(f(t)) \cdot f'(t)
\end{aligned}
$$

vector   scalar

we have used this before!

- If $F : \mathbb{R}^2 \to \mathbb{R}$ & $\vec{r} : \mathbb{R} \to \mathbb{R}^2$

then we also have a function $F \circ \vec{r} : \mathbb{R} \to \mathbb{R}$ defined by

$$
\begin{aligned}
(F \circ \vec{r})(t) &= F(\vec{r}(t)) \\
&= F(x(t), y(t))
\end{aligned}
$$

Example: $F(x,y)$ is the temperature at point $(x,y) \in \mathbb{R}^2$.

As you follow a path $\vec{r}(t) = (x(t), y(t))$ the temperature near you will change:

$$F(t) = F(\vec{r}(t)) = F(x(t), y(t)).$$

= the temperature that you feel at time t.

How does this temperature change?

$$F'(t) = ?$$

$$[F(\vec{r}(t))]' = ?$$

$$(F \circ \vec{r})'(t) = ?$$

What is the correct chain rule?

Multivariable Chain Rule:

Today we will show that

$$\boxed{[F(\vec{r}(t))]' = \nabla F(\vec{r}(t)) \cdot \vec{r}\,'(t)}$$

dot product of vectors

In other words, the relevant "derivative" of F is the gradient vector ∇F. We can also write this in terms of the components

$$\vec{r}(t) = (x(t), y(t))$$

to get

$$[F(x(t), y(t))]' = \nabla F \cdot \langle x'(t), y'(t) \rangle$$

$$= \left\langle \frac{dF}{dx}, \frac{dF}{dy} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

$$= \frac{dF}{dx} \cdot \frac{dx}{dt} + \frac{dF}{dy} \cdot \frac{dy}{dt}$$

or

$$\boxed{\frac{dF}{dt} = \frac{dF}{dx} \cdot \frac{dx}{dt} + \frac{dF}{dy} \cdot \frac{dy}{dt}}$$

More generally, if

$$\vec{r}(t) = (x_1(t), x_2(t), \dots, x_n(t))$$

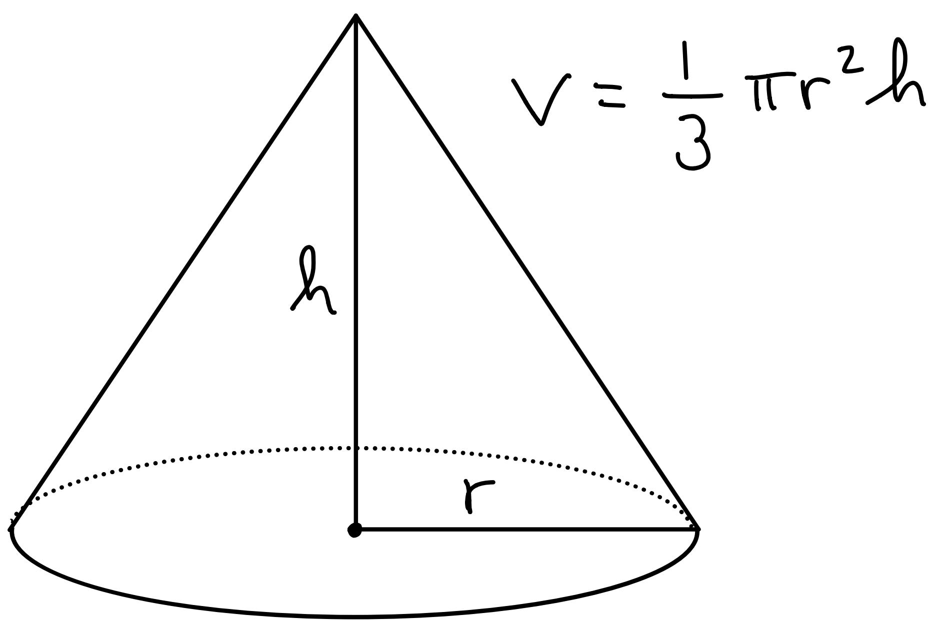
and $F : \mathbb{R}^n \to \mathbb{R}$ then

$$\boxed{\begin{gathered} \left[F(\vec{r}(t)) \right]' = \nabla F(\vec{r}(t)) \bullet \vec{r}\,'(t) \\ \frac{dF}{dt} = \frac{dF}{dx_1} \cdot \frac{dx_1}{dt} + \cdots + \frac{dF}{dx_n} \cdot \frac{dx_n}{dt} \end{gathered}}$$

Example: Consider a right circular cone of height h and radius r, which has volume

$$V(r, h) = \frac{1}{3} \pi r^2 h$$

Picture:



If $r(t)$ & $h(t)$ change with time then $V(t) = \frac{1}{3}\pi r(t)^2 h(t)$ also changes with time, and the rate of change is given by the multi-variable chain rule:

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} + \frac{dV}{dh} \cdot \frac{dh}{dt}$$

$$V'(t) = \left[\frac{2}{3}\pi r(t) h(t)\right] r'(t) + \left[\frac{1}{3}\pi r(t)^2\right] h'(t)$$

$$V' = \frac{2}{3}\pi r h r' + \frac{1}{3}\pi r^2 h'$$

Another point of view: Suppose we measure the radius & height with errors,

$$r = 120 \pm 1.8 \text{ in}$$

$$h = 140 \pm 2.5 \text{ in}$$

Then $V = \frac{1}{3}\pi r^2 h \pm ?$

$\approx 2111150 \pm ?$

Rewrite the chain rule in terms of "differential forms":

$$dV = \frac{2}{3}\pi r h \, dr + \frac{1}{3}\pi r^2 \, dh$$

$$\Delta V \approx \frac{2}{3}\pi r h \, \Delta r + \frac{1}{3}\pi r^2 \, \Delta h$$

In our case:

$r = 120$, $\Delta r = 1.8$

$h = 140$, $\Delta h = 2.5$

so that

$$\Delta V \approx \frac{2}{3}\pi (120)(140)(1.8) + \frac{1}{3}\pi (120)^2 (2.4)$$

$$\Delta V \approx 101033 \text{ in}^3$$

This is the approximate error

in the measurement of V :

$$V \approx 2.11 \pm 0.10 \text{ million in}^3$$

This is an approximation of an approximation, but it's good enough for most purposes.

The approximation is better when $\Delta r, \Delta h$ are small relative to r, h.

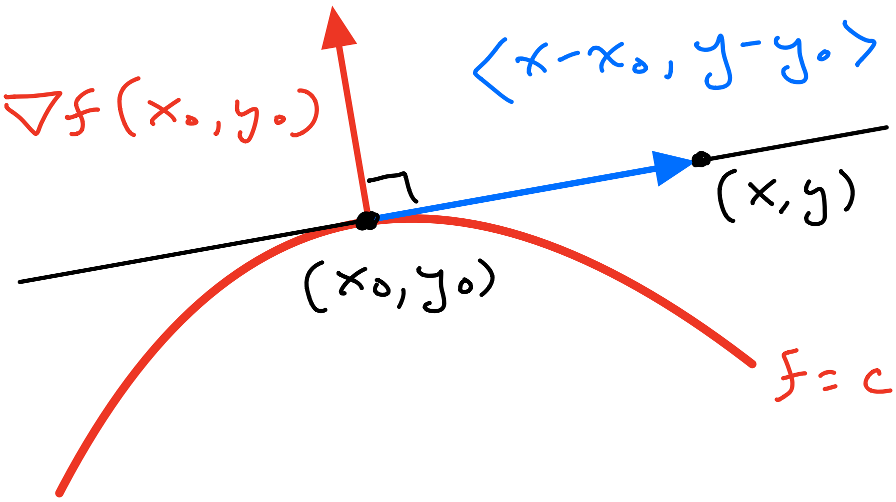
Why does it work ??

It has to do with tangent planes.

Recall : The tangent line to a curve $f(x,y) = \text{constant}$ at a point (x_0, y_0) has equation

$$\nabla f(x_0, y_0) \cdot \langle x - x_0, y - y_0 \rangle = 0$$

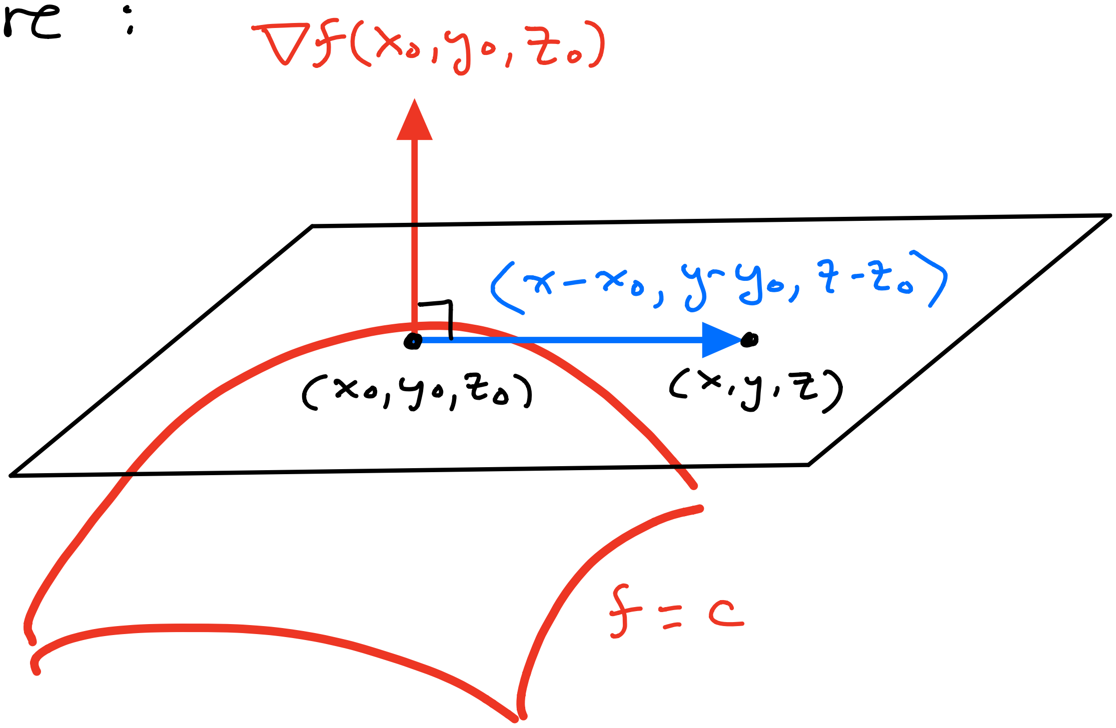
Picture :



Similarly, the tangent plane to a surface $f(x,y,z) = $ constant at point (x_0,y_0,z_0) has equation

$$\nabla f(x_0,y_0,z_0) \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$$

Picture :



Some surfaces in $\mathbb{R}^3$ have the form $z = f(x,y)$. Let's find the tangent plane to this surface at a point $(x_0, y_0, z_0) = (x_0, y_0, f(x_0,y_0))$.

Define $F(x,y,z) = f(x,y) - z$, so

$$z = f(x,y) \iff F(x,y,z) = 0$$

they define the same surface

Then we use the gradient vector formula for the tangent plane:

$$\nabla F(x_0, y_0, z_0) \bullet \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

Compute:

$$dF/dx = df/dx - 0$$

$$dF/dy = df/dy - 0$$

$$dF/dz = 0 - 1$$

so the tangent plane is

$$\left\langle \frac{df}{dx}, \frac{df}{dy}, -1 \right\rangle \bullet \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$\frac{df}{dx}(x - x_0) + \frac{df}{dy}(y - y_0) - 1(z - z_0) = 0$$

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

we can also write this as

$$dz = \frac{dz}{dx} \cdot dx + \frac{dz}{dy} \cdot dy ,$$

which is just the multivariable chain rule.

Pictures & Discussion next time!