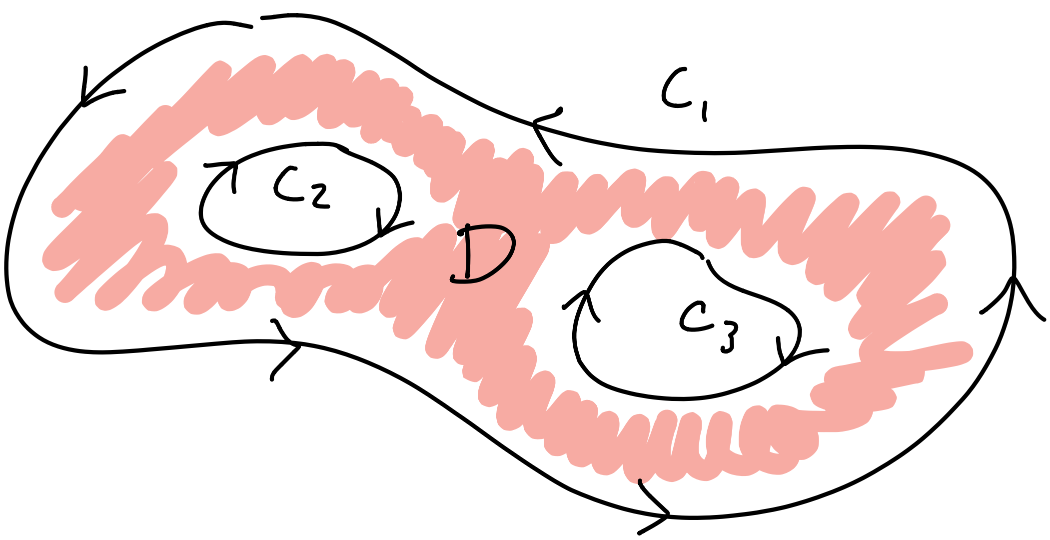
Final Project due Friday 11:59 PM

- Write a point form summary of the important definitions, formulas, theorems, pictures. Approx 10 pages.

Today: Bonus Content.

Recall Green's Theorem: Consider a vector field $\vec{F} = \langle P, Q \rangle$ and a 2D region D with oriented boundary curve ∂D:

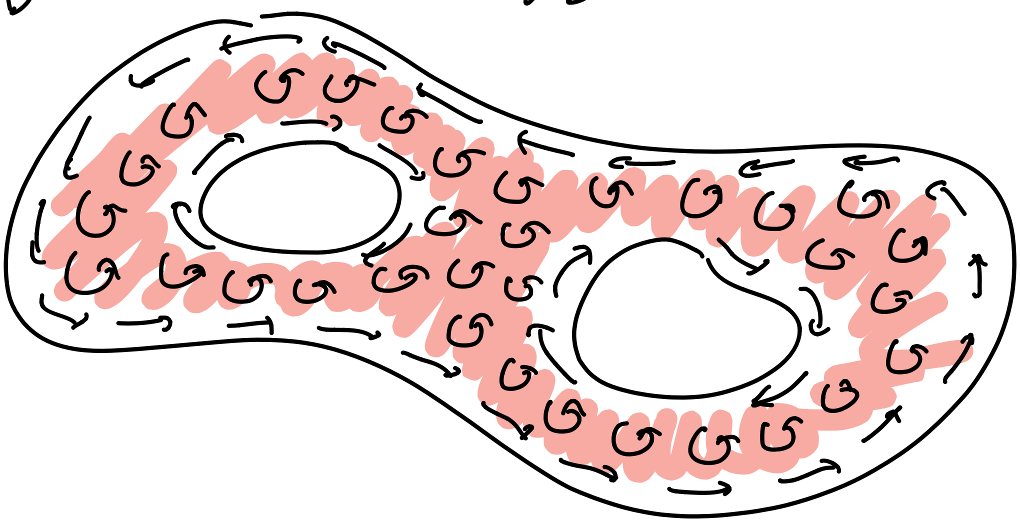


$$\partial D = C_1 + C_2 + C_2$$

[Orientation : D is "to the left" of ∂D.]

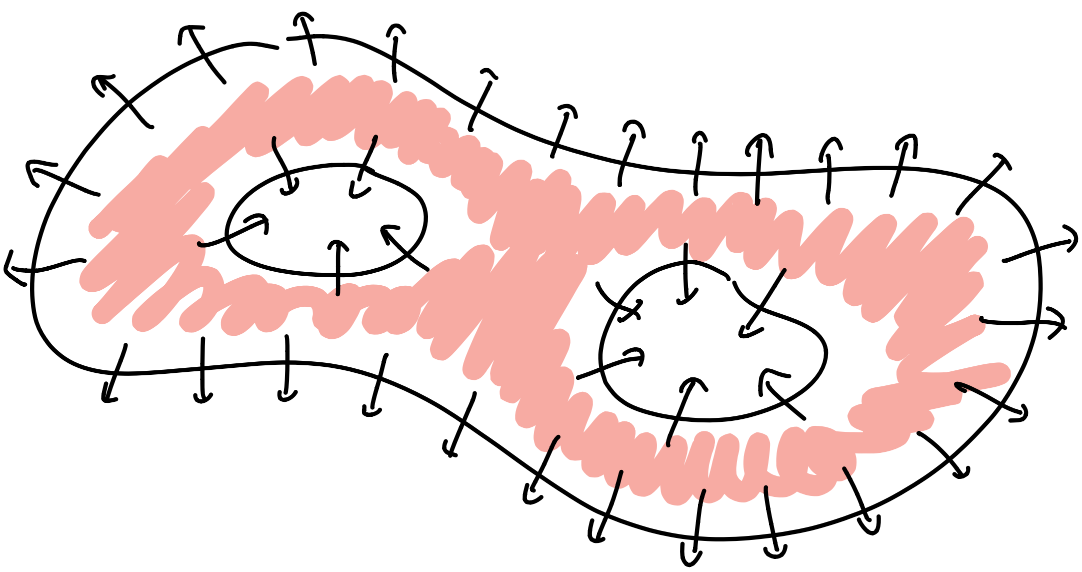
- Curl & Circulation:

$$\iint_D \operatorname{curl}(\vec{F})\, dA = \oint_{\partial D} \vec{F}\cdot\vec{T}\, ds$$



- Divergence & Flux :

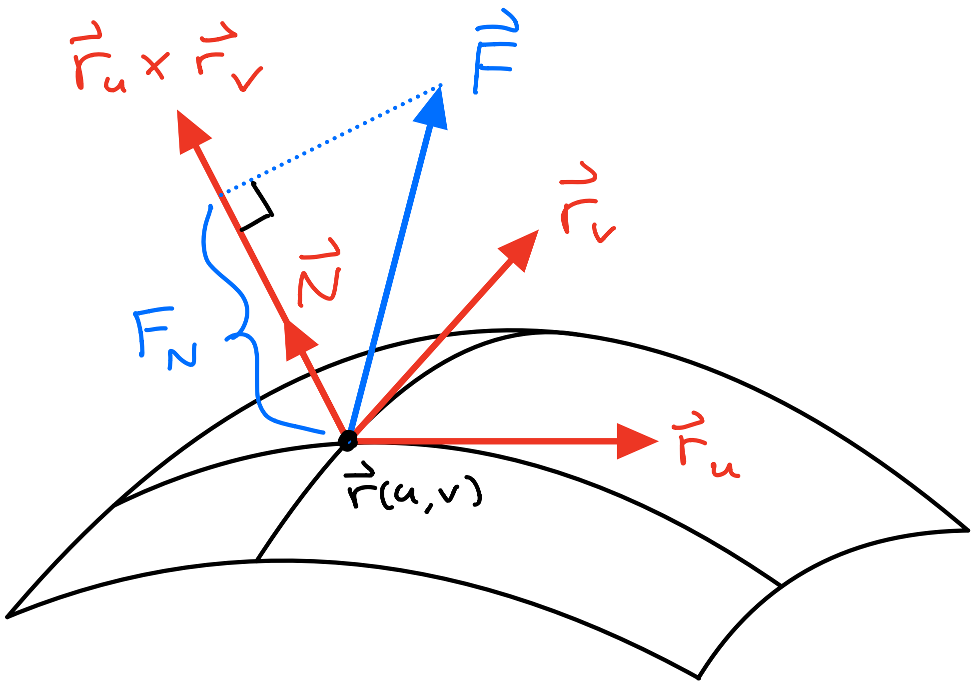
$$\iint_D \operatorname{div}(\vec{F})\, dA = \oint_{\partial D} \vec{F}\cdot\vec{N}\, ds$$



In 3D, each of these ideas is still correct but they become two different pictures. In order to state them we need to define the "flux of a vector field $\vec{F}$ across an oriented 2D surface D in $\mathbb{R}^3$":

$$\iint_D \vec{F} \cdot \vec{N} \, dA \quad (\text{What?})$$

Picture:



Given a parametrization $\vec{r}(u,v)$ of the surface D, the two "velocity vectors" $\vec{r}_u$ & $\vec{r}_v$ are tangent to the surface, so the cross product $\vec{r}_u \times \vec{r}_v$ is normal to the surface.

Consider a unit vector in the normal direction:

$$\vec{N} = \vec{r}_u \times \vec{r}_v \Big/ \| \vec{r}_u \times \vec{r}_v \|$$

Then the component of $\vec{F}$ in the normal direction is

$$F_N = \vec{F} \cdot \vec{N}$$

The flux of $\vec{F}$ across D is defined

$$\iint_D \vec{F} \cdot \vec{N} \, dA = \iint_D F_N \, dA$$

= how much is $\vec{F}$ pointing perpenicularly across the surface?

To compute it, recall that

$$dA = \underbrace{\|\vec{r}_u \times \vec{r}_v\| \, du\,dv}_{\text{tiny bit of surface area}},$$

so

$$\iint_D \vec{F} \cdot \vec{N} \, dA$$

$$= \iint \vec{F}(\vec{r}(u,v)) \cdot \frac{\vec{r}_u \times \vec{r}_v}{\cancel{\|\vec{r}_u \times \vec{r}_v\|}} \cancel{\|\vec{r}_u \times \vec{r}_v\|} \, du\,dv$$

$$= \iint \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \, du\,dv$$

[Remark: We assume $\vec{r}_u \times \vec{r}_v$ is never $\langle 0,0,0 \rangle$ so that $\vec{N}$ always exists and defines the "orientation" of D.]

For a vector field $\vec{F} = \langle P, Q, R \rangle$ in $\mathbb{R}^3$, recall the definition of curl

$$\nabla \times \vec{F} = \left\langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \right\rangle$$

and the definition of divergence

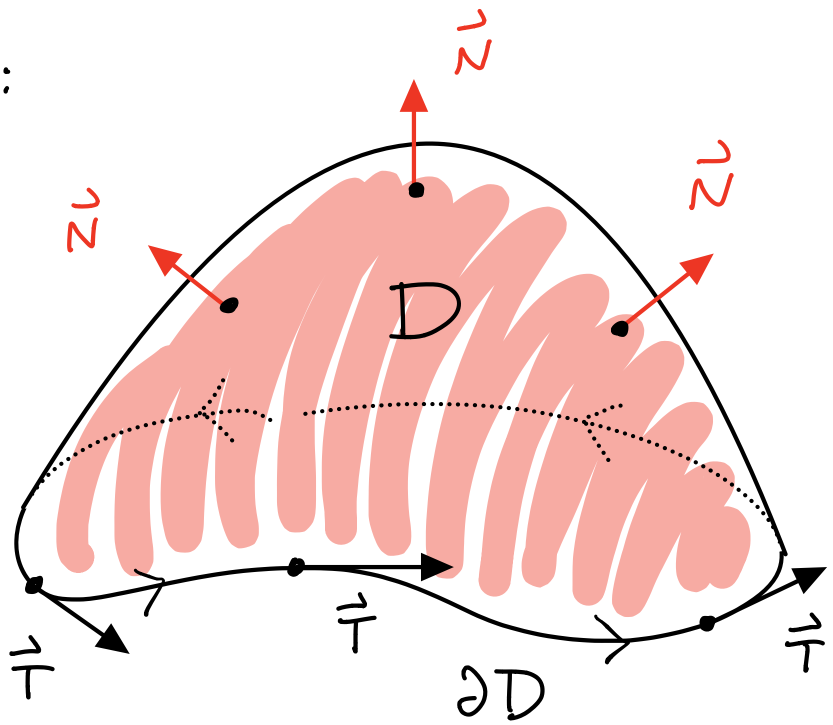
$$\nabla \cdot \vec{F} = P_x + Q_y + R_z$$

Stokes' Theorem: Let D be an oriented 2D surface in $\mathbb{R}^3$ with oriented boundary curve ∂D, which might have several pieces. Then

$$\iint_D (\nabla \times \vec{F}) \cdot \vec{N}\, dA = \oint_{\partial D} \vec{F} \cdot \vec{T}\, ds$$

total Flux of $\nabla \times \vec{F}$ across the surface D = total circulation of $\vec{F}$ along the curve ∂D.

Picture :



[Orientation : D is "to the left" of ∂D.]

Special Case : If D is a closed (orientable) surface, e.g., the surface of a sphere, then ∂D is nothing, and hence

$$\iint_D (\nabla \times \vec{F}) \cdot \vec{N} \, dA = 0 .$$

[Should remind you of conservative vector fields ...]

The Divergence Theorem:

Let E be a solid 3D region in $\mathbb{R}^3$ and let ∂E be the "oriented boundary surface" of E.

[Orientation: The normal vector $\vec{N}$ of the surface ∂E points "out of" E.]

Then:

$$\iiint_E (\nabla \circ \vec{F})\, dV = \iint_{\partial E} \vec{F} \circ \vec{N}\, dA$$

integral of the scalar field $\nabla \circ \vec{F}$ over the 3D region E = flux of the vector field $\vec{F}$ <u>across</u> the 2D surface ∂E.

Intuition (Fluid Dynamics):

If $\vec{F}$ is the velocity field of a fluid,

$$\iiint_E (\nabla \cdot \vec{F})\, dV = \iint_{\partial E} \vec{F} \cdot \vec{N}\, dA$$

how much does the fluid expand/contract inside the region E? = how much does the fluid flow across the boundary ∂E?

That makes sense!

Next Time: More bonus content on the physical applications of the fundamental theorems.