HW 5 will be posted tomorrow, due Tuesday.

Review:

- To integrate a scalar field f along an oriented curve C:

$$\int_C f\, ds = \int f(\vec{r}(t))\, \|\vec{r}\,'(t)\|\, dt$$

Example: Let $f(x,y) = x$ be the height of a wall above point (x,y). If the base of the wall is the curve $y = x^2$ $(0 \le x \le 1)$, find the area of the wall.

To compute this we must parametrize the curve:

$$\vec{r}(t) = \langle t, t^2 \rangle \text{ from } t=0 \text{ to } t=1.$$

Then :

$$\text{area of wall} = \int \text{area of skinny rectangle}$$

$$= \int \underset{\text{height}}{f}\ \underset{\text{length of base}}{ds}$$

$$= \int_0^1 f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$$

$$= \int_0^1 t\sqrt{1^2+(2t)^2}\,dt$$

$$= \int_0^1 t\sqrt{1+4t^2}\,dt$$

$$\left[\, u = 1+4t^2, \quad du = 8t\,dt \,\right]$$

$$= \frac{1}{8}\int_1^5 \sqrt{u}\,du$$

$$= \frac{1}{8}\left(\frac{2}{3}u^{3/2}\right)\Big|_1^5$$

$$= \frac{1}{8}\left(\frac{2}{3}\cdot 5^{3/2} - \frac{2}{3}\right)$$

$$\approx 0.85$$

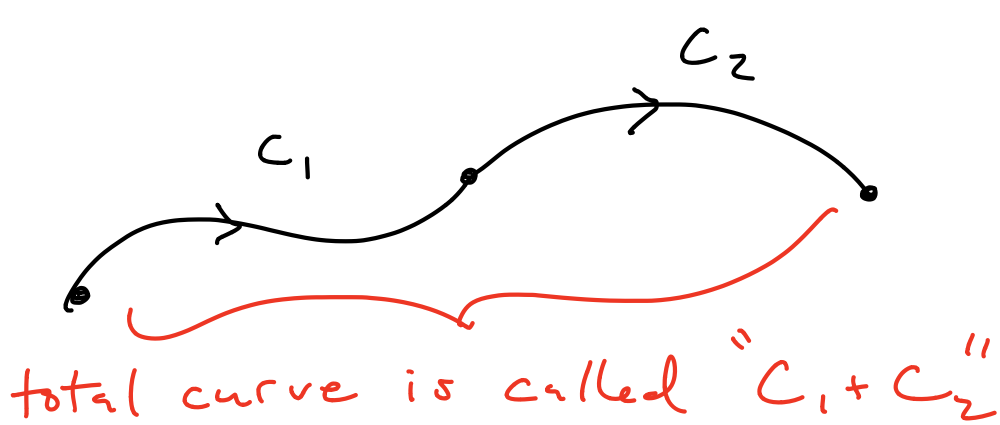
- To integrate a vector field $\vec{F}$ along an oriented curve C:

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

Example: Your increase in kinetic energy (KE) due to the force $\vec{F}$.

- Reversing & Concatenating Curves:

Curves can be "added & subtracted"



The reverse orientation of a curve C is called "$-C$".

Theorem:

$$\text{``}\int_{C_1+C_2}\text{''} = \text{``}\int_{C_1}\text{''} + \text{``}\int_{C_2}\text{''}$$

$$\text{``}\int_{-C}\text{''} = \text{``}-\int_{C}\text{''}$$

[Compare: $\text{``}\int_a^b\text{''} = \text{``}-\int_b^a\text{''}$.]

- Fundamental Theorem of Line Integrals:

$$\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = f(\vec{r}(b)) - f(\vec{r}(a))$$

[does not depend on the "shape" of the curve; only the endpoints.]

In particular, if you follow a loop ($\vec{r}(a) = \vec{r}(b)$) then

$$\int_{\text{loop}} \nabla f \cdot \vec{T} \, ds = 0$$

- Conservation of Energy:

If $\vec{F} = -\nabla f$ then we can think of $\vec{F}$ as a "force field" and f as the "potential energy" (PE).

[The force is trying to decrease your potential energy!]

Then

$$\int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = -\left[f(\vec{r}(b)) - f(\vec{r}(a))\right]$$

increase in KE = decrease in PE

In other words, the "total

mechanical engergy" KE + PE

is conserved over time.

For this reason, a vector field

of the form $\vec{F} = \nabla f$ (or $-\nabla f$)

is called "conservative".

- Not all vector fields are conservative.

Example: Let $\vec{F}(x,y) = \langle -y, x \rangle$.

We will show that $\vec{F}$ is not cons.

by finding a loop C such that

$$\int_C \vec{F} \cdot \vec{T}\, ds \neq 0$$

Here's the loop:

$\vec{r}(t) = \langle \cos t, \sin t \rangle$, $t = 0$ to 2π

Then

$$\int \vec{F} \cdot \vec{T}\, ds = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t) \, dt$$

$$= \int_0^{2\pi} 1 \, dt = 2\pi \neq 0 .$$

Conclusion: There is <u>no</u> function $f(x,y)$ such that

$$\vec{F} = \nabla f$$

$$\langle -y, x \rangle = \langle f_x , f_y \rangle .$$

Conversely, if $\vec{F}$ is any vector field satisfying

$$\int_{\text{Loop}} \vec{F} \cdot \vec{T} \, ds = 0$$

for *every* loop, then I claim that $\vec{F}$ is conservative, i.e., there exists a scalar field f such that

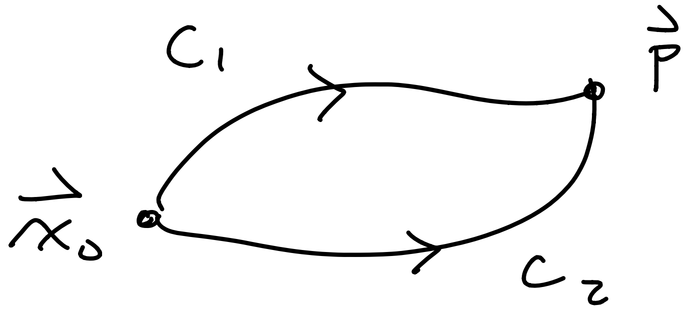
$$\vec{F} = \nabla f.$$

Idea: Fix an arbitrary basepoint $\vec{x}_0$. Then for any point $\vec{p}$ we define

$$f(\vec{p}) = \int_{\vec{x}_0}^{\vec{p}} \vec{F} \cdot \vec{T}\, ds,$$

where the integral is taken along *any path* from $\vec{x}_0$ to $\vec{p}$.

To see that this makes sense, consider two different paths



Then "$C_1 - C_2$" is a <u>loop</u> so that

$$\int_{C_1 - C_2} \vec{F} \cdot \vec{T}\, ds = 0$$

$$\int_{C_1} \vec{F} \cdot \vec{T}\, ds - \int_{C_2} \vec{F} \cdot \vec{T}\, ds = 0$$

$$\int_{C_1} \vec{F} \cdot \vec{T}\, ds = \int_{C_2} \vec{F} \cdot \vec{T}\, ds \quad \checkmark$$

[Recall from Calc I:

$$\frac{d}{dx} \int_{x_0}^{x} f(t)\, dt = f(x). \text{]}$$

Unfortunately, it is very hard to check that $\int \vec{F} \cdot \vec{T}\, ds = 0$ around <u>any</u> loop. [There are infinitely many possible loops!]

It turns out there is an easier way to determine if a vector field is conservative.

Idea: If $\vec{F} = \nabla f$ and f represents "height" then $\vec{F}$ points "uphill".

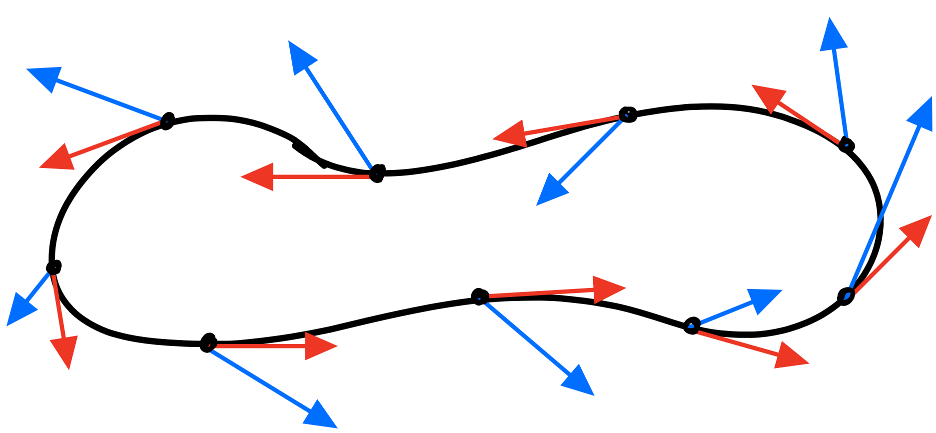
If $\displaystyle\int_{\text{loop}} \vec{F} \cdot \vec{T} \, ds \neq 0$ (say > 0)

then this is a contradiction because you walked around a loop but you ended up "higher" than you started.

For a general field $\vec{F}$ we get

$\int \vec{F} \cdot \vec{T} \, ds > 0$ when $\vec{F}$ (on average)

points in the direction $\vec{T}$ of the loop:



This "curling" of a vector field F around a loop can be precisely measured by the "curl" $\nabla \times \vec{F}$.

[Warning: This concept is only defined in $\mathbb{R}^3$, and partially in $\mathbb{R}^2$.]

Given a vector field $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$,

$$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle,$$

we define the "curl"

$$\nabla \times \vec{F} = \text{"} \langle \partial_x, \partial_y, \partial_z \rangle \times \langle P, Q, R \rangle \text{"}$$

$$= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

Theorem (Cross-Partial Property of Conservative Vector Fields):

- Given a vector field $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$,

$\vec{F}$ is conservative $\iff$ $\nabla \times \vec{F} = \langle 0,0,0 \rangle$

$$\iff \begin{cases} R_y = Q_z \\ P_z = R_x \\ Q_x = P_y \end{cases}$$

- Given a vector field $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ written as $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$,

$\vec{F}$ is conservative $\iff$ $Q_x = P_y$

Remarks: These criteria are very easy to check! :)

Example: Prove that the vector field

$$\vec{F}(x,y,z) = \langle \underbrace{3x^2 z}_{P}, \underbrace{z^2}_{Q}, \underbrace{x^3 + 2yz}_{R} \rangle$$

is conservative.

Proof : Check the "cross-partials"

$$R_y = 0 + 2z \,, \quad Q_z = 2z \quad \checkmark$$

$$P_z = 3x^2 \,, \quad R_x = 3x^2 + 0 \quad \checkmark$$

$$Q_x = 0 \,, \quad P_y = 0 \quad \checkmark$$

Done. ///

This proves that an "antiderivative" $f(x,y,z)$ <u>exists</u> such that

$$\vec{F} = \nabla f$$

but it does not tell us *how* to <u>find</u> f.

In this case I can tell you that

$$f(x,y,z) = x^3 z + y z^2 + \text{constant}$$

because I planned it that way!

In general it is <u>*hard*</u> to find antiderivatives.