HW5 : Tues June 22

Quiz 5 : Wed June 23

Last time we discussed the integral of a scalar field f along a curve C:

$$\int_C f \, \underbrace{ds}_{\text{tiny amount of length}}$$

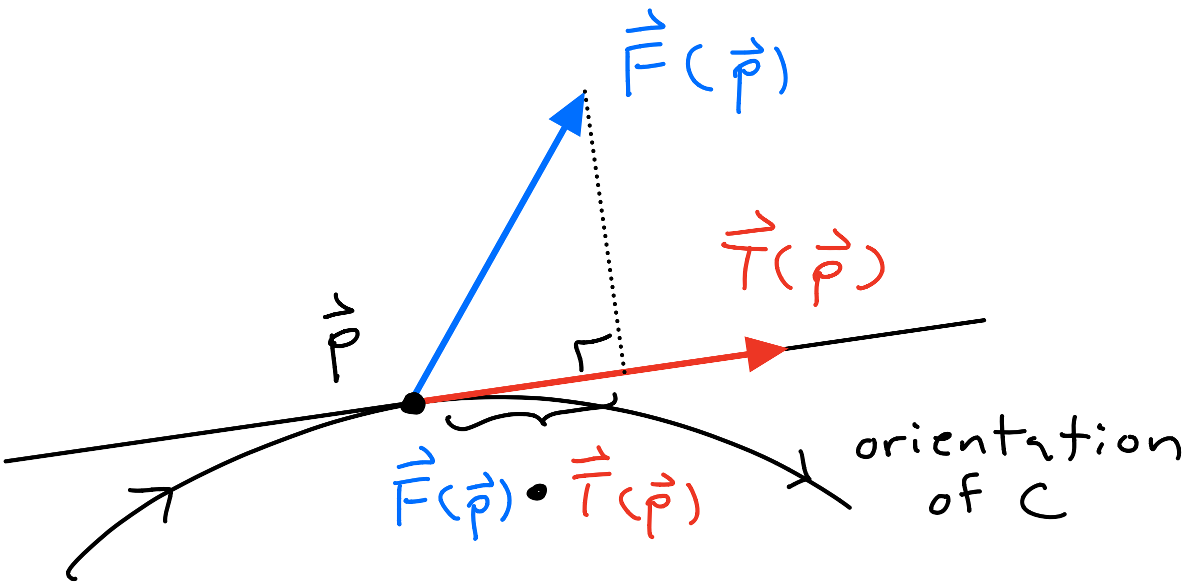
[Example: If $f = 1$ then this is just the arc length of C.]

To actually <u>compute</u> this integral we must choose a parametrization $\vec{r}(t)$ of the curve C. Then:

$$\int_C f \, ds = \int f(\vec{r}(t)) \, \|\vec{r}\,'(t)\| \, dt$$

We also discussed the integral of a vector field $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ along an oriented curve C in $\mathbb{R}^n$.

Idea: At every point $\vec{p}$ of C there is a unit (length 1) tangent vector $\vec{T}(\vec{p})$ in the direction of the orientation:



The component of the vector $\vec{F}(\vec{p})$ in the direction of the tangent $\vec{T}(\vec{p})$ is the dot product $\vec{F}(\vec{p}) \cdot \vec{T}(\vec{p})$, which may be negative. Then the "integral of $\vec{F}$ along C" is defined to be

$$\int_C \vec{F} \cdot \vec{T} \, ds = \text{how much is } \vec{F} \text{ pointing in the direction of } \vec{T}?$$

To actually <u>compute</u> this, we must choose a parametrization $\vec{r}(t)$ for C, agreeing with the orientation. Then

$$\vec{T}(\vec{r}(t)) = \vec{r}\,'(t) / \|\vec{r}\,'(t)\| ,$$

$$ds = \|\vec{r}\,'(t)\| \, dt ,$$

so that

$$\int_C \vec{F} \cdot \vec{T} \, ds = \int \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{\cancel{\|\vec{r}\,'(t)\|}} \cancel{\|\vec{r}\,'(t)\|} \, dt$$

$$= \int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt .$$

Okay, but why do we care?

Suppose $\vec{F}$ is a force acting on you as you travel a path $\vec{r}(t)$.

Then

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \text{force} \cdot \text{velocity}$$

is the "work done on you by the force", and the integral

$$\int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int \text{work}$$

is the "total kinetic energy imparted to you by the force".

Example: A frictional force always opposes your motion, so that

$$\vec{F}_{friction}(\vec{r}(t)) \cdot \vec{r}\,'(t) < 0.$$

Over time this force will drain your kinetic energy:

$$\underbrace{\int \vec{F}_{friction} \cdot d\vec{r}}_{\text{change in your kinetic energy}} < 0$$

What about gravity?

Important Observation:

The gravitational force field $\vec{F}$ due to a single mass (like the sun) has an "antiderivative" scalar field f called the "gravitational potential":

$$\vec{F} = \nabla f$$

gravitational force

gravitational potential

Details: If the sun is fixed at $(0,0,0)$ and a planet is at

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle,$$

then the planet feels a force of magnitude $1/\|\vec{r}(t)\|^2$, pointed directly towards $(0,0,0)$:

$$\vec{F}(\vec{r}(t)) = -\frac{1}{\|\vec{r}(t)\|^2} \frac{\vec{r}(t)}{\|\vec{r}(t)\|}$$

$$= -\frac{1}{\|\vec{r}(t)\|^3} \vec{r}(t)$$

$$= -\frac{1}{(x^2+y^2+z^2)^{3/2}} \langle x, y, z \rangle$$

$$= \left\langle \frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

Does this vector field have an "anti-derivative"? I claim that

$$\vec{F} = -\nabla f$$

negative sign is a physics convention

where

$$f(\vec{r}(t)) = -1/\|\vec{r}(t)\|$$

$$= -(x^2+y^2+z^2)^{-1/2}$$

Proof: We have

$$f_x = \frac{d}{dx}(x^2+y^2+z^2)^{-1/2}$$

$$= \left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-3/2}(2x)$$

$$= -x/(x^2+y^2+z^2)^{3/2} \quad \checkmark$$

Same calculation for f_y & f_z. ///

This seemingly boring fact has an amazing cosequence.

Conservation of Energy:

Suppose the planet follows a path $\vec{r}(t)$ from $t=a$ to $t=b$. Then its change in Kinetic energy due to gravity is

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= -\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

multvariable chain rule

$$= -\int_a^b \left(f(\vec{r}(t)) \right)' dt$$

Calc I

$$= -\left[f(\vec{r}(b)) - f(\vec{r}(a)) \right]$$

Meaning : The increase in KE does not depend on the shape of the path ; only on the endpoints $\vec{r}(a)$ & $\vec{r}(b)$. In fact, since

$$f(\vec{r}(t)) = -1 \Big/ \|\vec{r}(t)\|,$$

it only depends on the "heights" $\|\vec{r}(a)\|$ & $\|\vec{r}(b)\|$ "above" the sun.

In other words :

increase in KE = decrease in PE

= how far "downhill" did the planet go ?

This is one of the most important concepts in physics.

The Fundamental Theorem for Line Integrals :

Suppose a vector field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ has an "antiderivative scalar field" $f : \mathbb{R}^n \to \mathbb{R}$ so that

$$\vec{F} = \nabla f ,$$

in which case we say that $\vec{F}$ is a "conservative vector field". Then the integral of $\vec{F}$ along a path $\vec{r}(t)$ from $t = a$ to $t = b$ only depends on the endpoints :

$$\boxed{\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = f(\vec{r}(b)) - f(\vec{r}(a))}$$

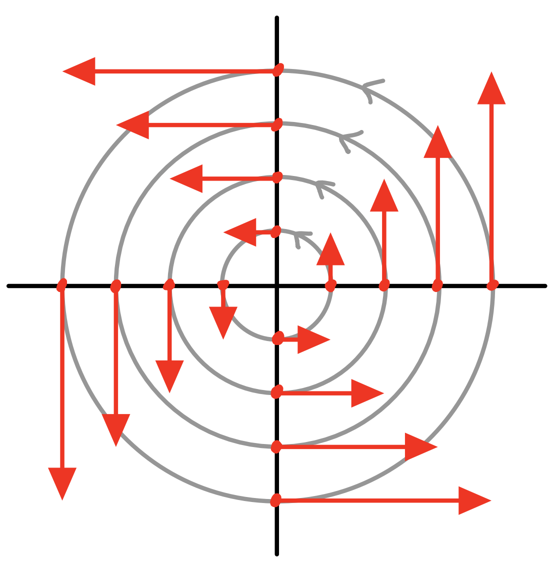
If the path goes around a loop, so $\vec{r}(a) = \vec{r}(b)$, then we get

$$\int_{\text{loop}} \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = 0$$

Very interesting!

We have seen that gravity is a conservative force. But not every force field is conservative.

Example: Consider $\vec{F}(x,y) = \langle -y, x \rangle$



Consider the loop $\vec{r}(t) = \langle \cos t, \sin t \rangle$ from $t = 0$ to $t = 2\pi$. The integral of $\vec{F}$ along $\vec{r}$ (if $\vec{F}$ is a force, this is the change in KE) is

$$\int_0^{2\pi} \vec{F}(\vec{r}(t)) \circ \vec{r}\,'(t)\, dt$$

$$= \int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt$$

$$= \int_0^{2\pi} 1\, dt = 2\pi \neq 0$$

It follows that $\vec{F}(x,y) = \langle -y, x \rangle$ is not conservative, i.e., there does not exist a scalar field

$f(x,y)$ such that $\vec{F} = \nabla f$.

Intuition: If $\vec{F} = \nabla f$ is a gradient field where f represents "height", then $\vec{F}$ always points "uphill". But there is no such thing as a loop that always goes uphill!