Review for Quiz 4.

Coordinate systems in 2D:

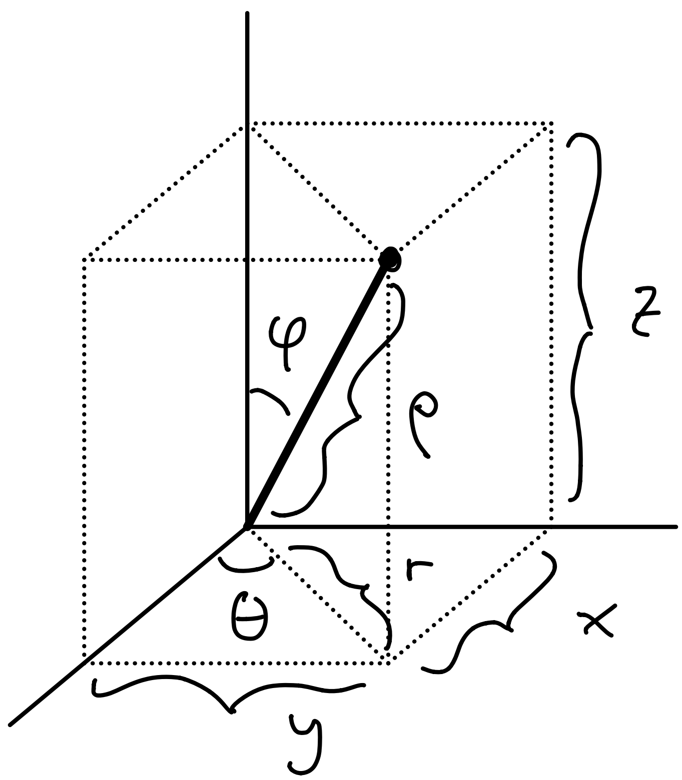
Cartesian vs. Polar

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = y/x \end{cases}$$

$$dA = dx\,dy = r\,dr\,d\theta$$

Use polar coordinates when the domain or the integrand has rotational symmetry.

Coordinate systems in 3D:



Cartesian vs. Cylindrical

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz$$

Cylindrical vs. Spherical

$$\begin{cases} r = \rho \sin\varphi \\ z = \rho\cos\varphi \end{cases} \qquad \begin{cases} \rho^2 = r^2 + z^2 \\ \tan\varphi = r/z \end{cases}$$

Cartesian vs. Spherical

$$\begin{cases} x = r\cos\theta = \rho\sin\varphi\cos\theta \\ y = r\sin\theta = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$$

$$\begin{cases} \rho^2 = r^2 + z^2 = x^2 + y^2 + z^2 \\ \tan\varphi = r/z = \sqrt{x^2+y^2}/z \\ \tan\theta = y/x \end{cases}$$

The volume element in spherical coords:

$$dV = \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$$

Examples :

- Parametrize the tetrahedron

$x, y, z \geq 0$ & $x + 2y + z \leq 1$.

Let $y = z = 0$. Then

$$0 \leq x \leq 1$$

Let $z = 0$. Then

$$x + 2y + 0 \leq 1$$
$$0 \leq y \leq (1-x)/2$$

Finally, we have

$$x + 2y + z \leq 1$$
$$0 \leq z \leq 1 - 2y - x$$

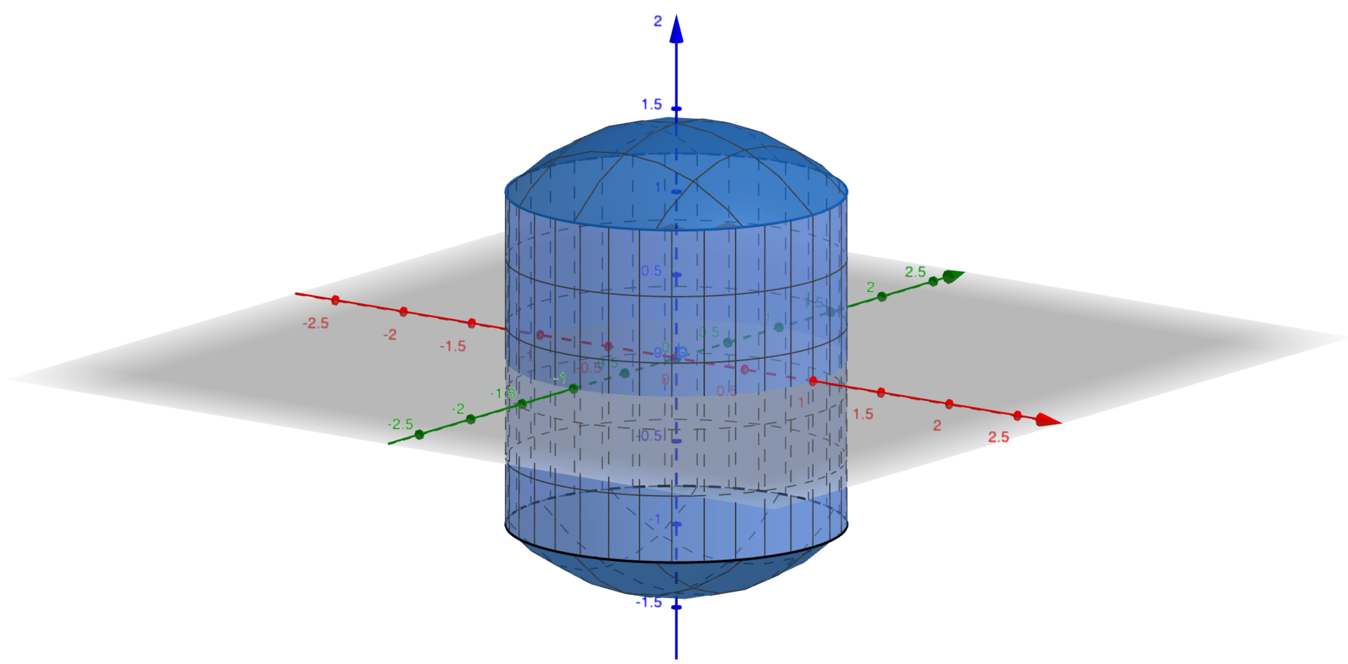
The integral of any scalar field $f(x,y,z)$ over the tetrahedron can be computed as follows:

$$\iiint f \, dV$$

$$= \int_0^1 \left(\int_0^{(1-x)/2} \left(\int_0^{1-2y-x} f \, dz \right) dy \right) dx$$

- Parametrize the solid region that is the intersection of the sphere $x^2+y^2+z^2 \leq 2$ and the cylinder $x^2+y^2 \leq 1$.

Picture :



We should use cylindrical coords :

$$0 \le r \le 1$$

$$0 \le \theta \le 2\pi$$

$$-\sqrt{2-x^2-y^2} \le z \le +\sqrt{2-x^2-y^2}$$

top & bottom of sphere

The formula involves $r^2 = x^2+y^2$:

$$-\sqrt{2-r^2} \le z \le +\sqrt{2-r^2}$$

[This happens whenever the problem has rotational symmetry around the z-axis.]

Let's compute the volume :

$$\text{vol} = \iiint dV$$

$$= \iiint r\,dr\,d\theta\,dz$$

$$= \int_0^{2\pi} d\theta \int_0^1 r \int_{-\sqrt{2-r^2}}^{+\sqrt{2-r^2}} dz\, dr$$

$$= 2\pi \int_0^1 2r\sqrt{2-r^2}\, dr$$

$[\text{let } u = 2-r^2,\ du = -2r\,dr]$

$$= 2\pi \int_2^1 -\sqrt{u}\, du$$

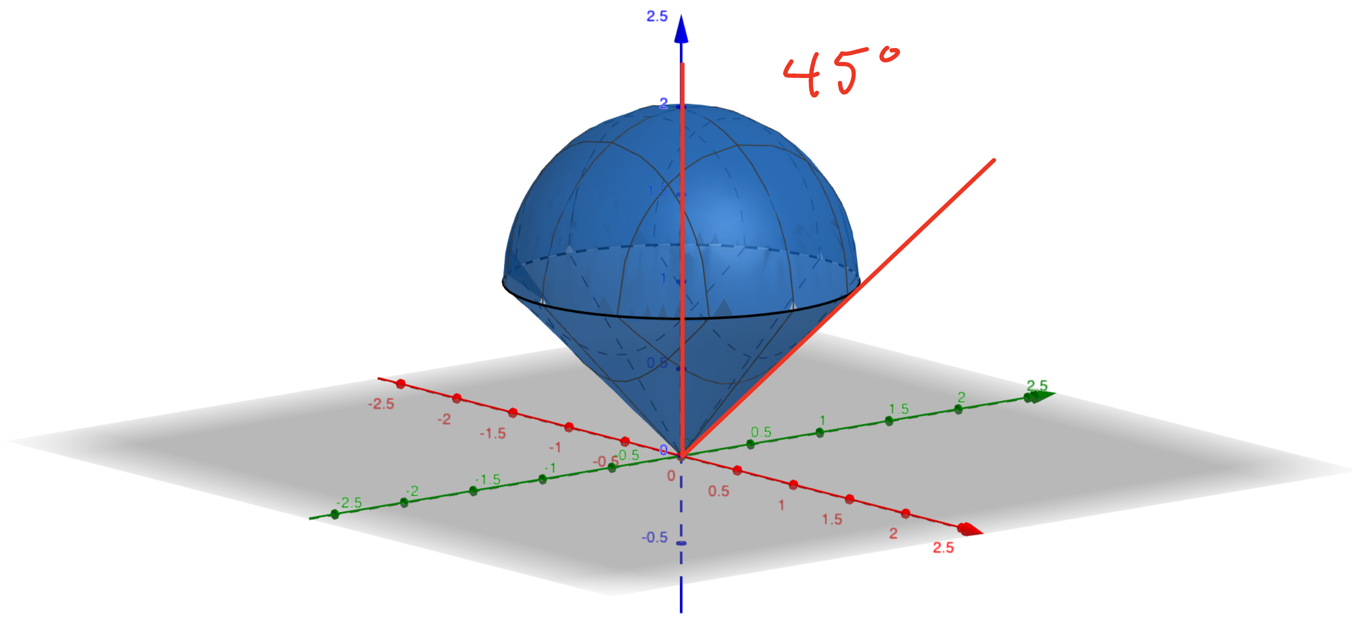
$$= 2\pi \left(-\frac{1}{3/2} u^{3/2} \right)_2^1$$

$$= 2\pi \left(\frac{4\sqrt{2}}{3} - \frac{2}{3} \right)$$

$$\approx (1.22)\, 2\pi$$

The caps add about 22% to the volume of the cylinder.

• Parametrize the "ice cream cone" between the cone $x^2+y^2=z^2$ and the sphere $x^2+y^2+(z-1)^2=1$:



Let's use spherical coords:

$$0 \le \theta \le 2\pi$$

$$0 \le \varphi \le \pi/4$$

$$0 \le \rho \le ?$$

The value of ρ depends on the equation of the sphere:

$$x^2 + y^2 + (z-1)^2 \leq 1$$

$$x^2 + y^2 + z^2 - 2z + 1 \leq 1$$

$$x^2 + y^2 + z^2 \leq 2z$$

$$\rho^2 \leq 2\rho\cos\varphi$$

$$\rho \leq 2\cos\varphi \quad \checkmark$$

To integrate a scalar field $f(r, \theta, \varphi)$ over the ice cream cone:

$$\iiint f \, dV$$

$$= \iiint f \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\varphi \left(\int_0^{2\cos\varphi} f \rho^2 \, d\rho \right) d\varphi$$