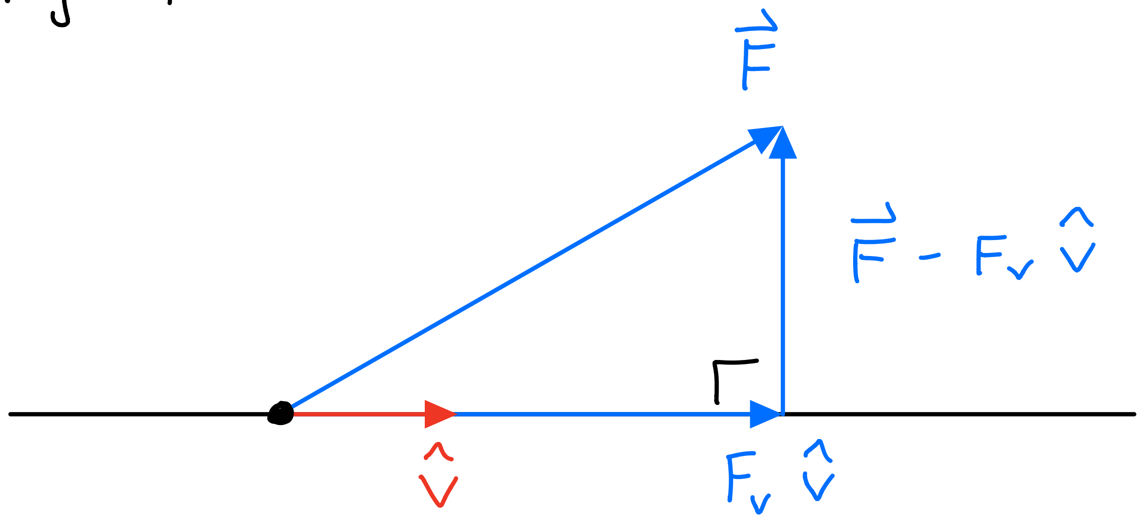
Homework 5 Solutions & Discussion :

1. Projection :



Let $F_v \hat{v}$ be the projection of $\vec{F}$ onto the line spanned by a unit vector $\hat{v} = \vec{v}/\|\vec{v}\|$. Since $\|\hat{v}\| = 1$ we have

$$\| F_v \hat{v} \| = | F_v | .$$

To compute the scalar F_v, we use the fact (which is true by definition) that the vectors $\vec{F} - F_v \hat{v}$ and $\hat{v}$ are perpendicular, so their dot product is equal to zero :

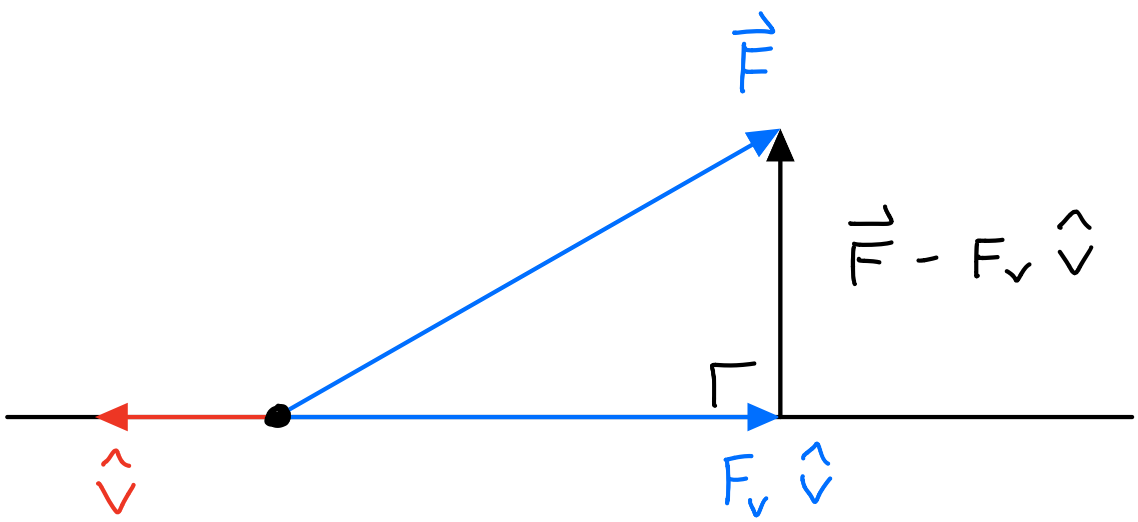
$$(\vec{F} - F_v \hat{v}) \cdot \hat{v} = 0$$

$$(\vec{F} \cdot \hat{v}) - F_v \underbrace{(\hat{v} \cdot \hat{v})}_{1} = 0$$

$$\vec{F} \cdot \hat{v} = F_v$$

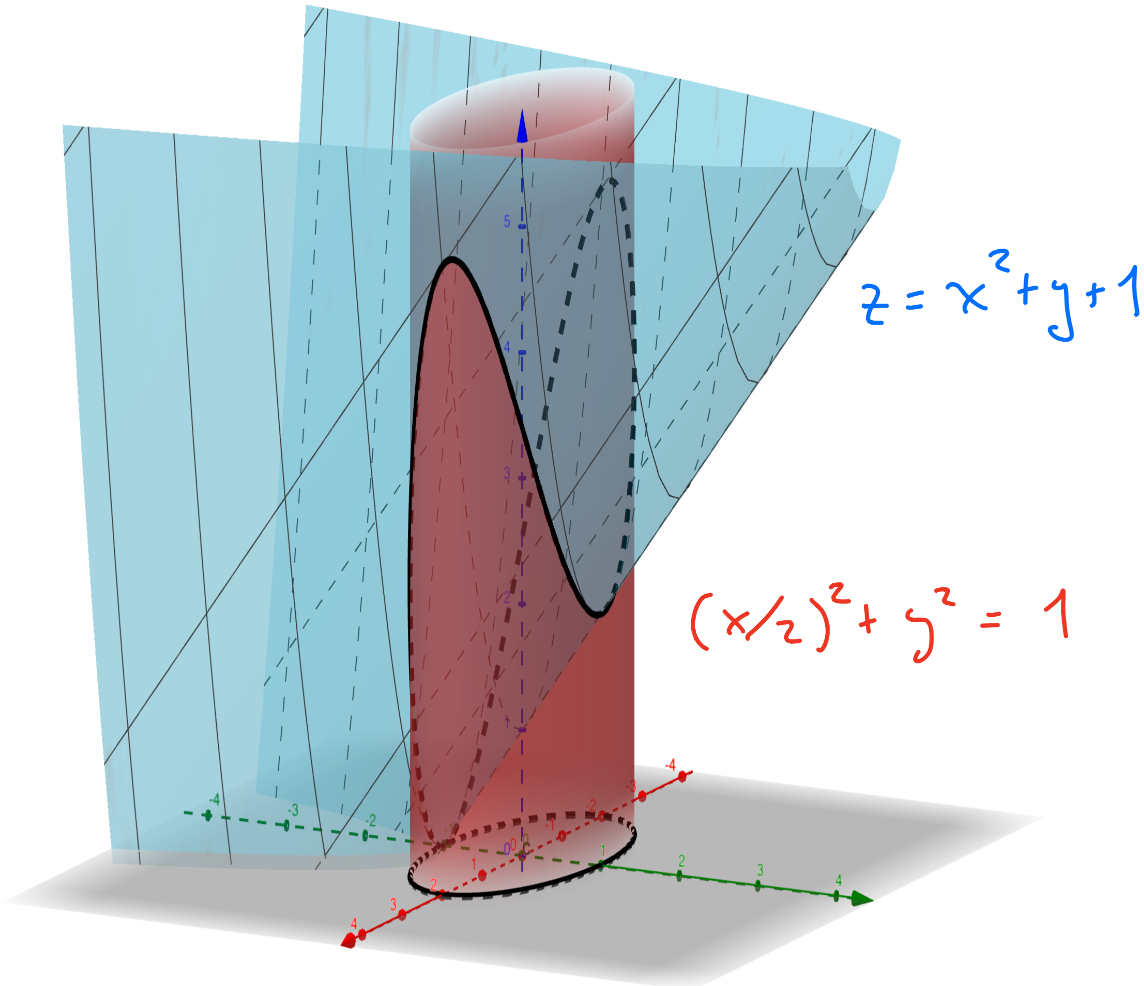
Done!

Note that the scalar F_v might be negative, as in the following picture:



2. Find the area of the wall above the ellipse $(x/2)^2 + y^2 = 1$ and below the surface $z = x^2 + y + 1$.

Here is a picture:



[The wall is the red surface between the two black curves.]

To compute the area we parametrize the ellipse as follows:

$$\vec{r}(t) = \langle 2\cos t, \sin t \rangle$$

$$\vec{r}\,'(t) = \langle -2\sin t, \cos t \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{4\sin^2 t + \cos^2 t}$$

Then

$$\text{Area} = \int \text{area of skinny rectangle}$$

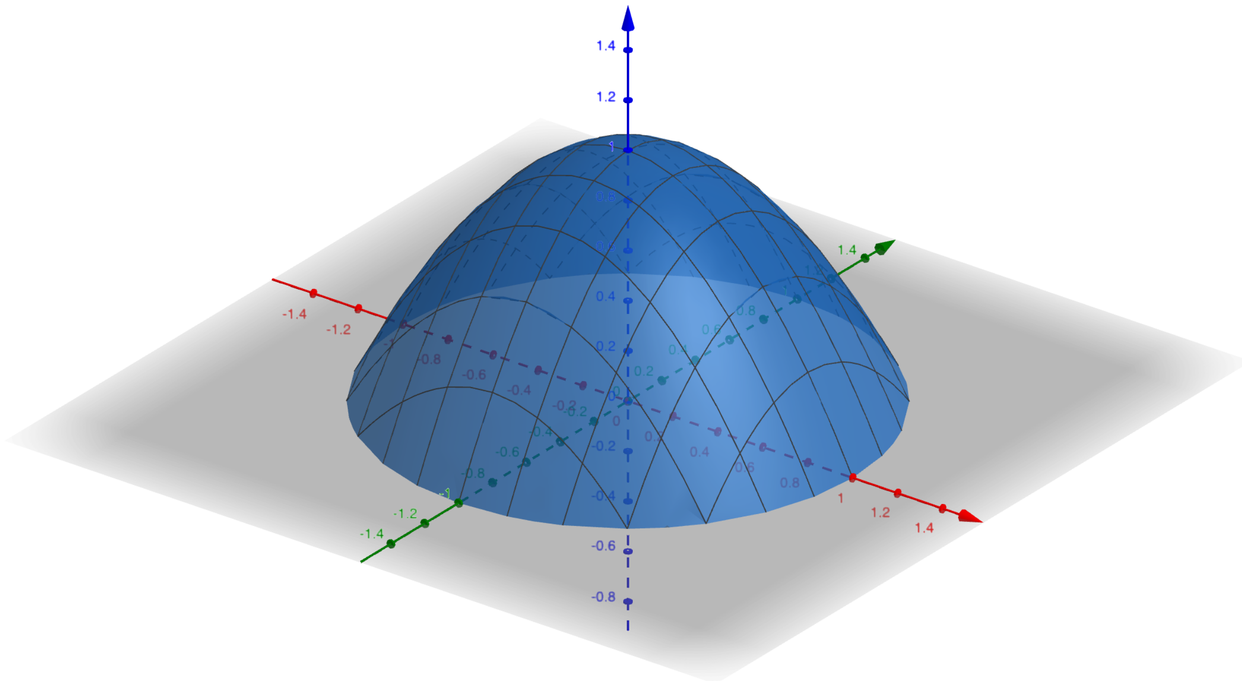
$$= \int \underbrace{\left((2\cos t)^2 + \sin t + 1 \right)}_{\text{height}} \underbrace{\|\vec{r}\,'(t)\|}_{\text{length of base}} \, dt$$

$$= \int_0^{2\pi} \left(4\cos^2 t + \sin t + 1 \right) \sqrt{4\sin^2 t + \cos^2 t} \; dt$$

$\vdots$ computer

$$\approx 25.968$$

3. Find the area of the top of the parabolic dome $z = 1 - x^2 - y^2$, where $x^2 + y^2 \leq 1$:



We parametrize the surface using polar coordinates:

$$x = u \cos v$$
$$y = u \sin v$$
$$z = 1 - x^2 - y^2 = 1 - u^2$$

$$\vec{r}(u,v) = \langle u \cos v, u \sin v, 1-u^2 \rangle$$

$$\vec{r}_u = \langle \cos v, \sin v, -2u \rangle$$

$$\vec{r}_v = \langle -u \sin v, u \cos v, 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 2u^2 \cos v, 2u^2 \sin v, u \rangle$$

$$\| \vec{r}_u \times \vec{r}_v \| = \sqrt{4u^4 + u^2}$$

$$= u\sqrt{4u^2 + 1}$$

So the surface area is

$$\text{Area} = \iint dS$$

$$= \iint \underbrace{\| \vec{r}_u \times \vec{r}_v \| \, du\, dv}_{\text{area of tiny parallelogram on the surface}}$$

$$= \iint u\sqrt{4u^2+1} \, du\, dv$$

$$= \int_0^{2\pi} dv \int_0^1 u\sqrt{4u^2+1} \, du$$

[Let $w = 4u^2 + 1$, $dw = 8u\, du$]

$$= \int_0^{2\pi} dv \int_1^5 \frac{1}{8}\sqrt{w}\, dw$$

$$= 2\pi \cdot \frac{1}{8}\left(\frac{2}{3} w^{3/2}\right)_1^5$$

$$= \frac{\pi}{6}\left(5^{3/2} - 1\right)$$

$$\approx 2\pi (0.85)$$

[About 15% less than the area of the hemisphere of radius 1.]

Problem 4. Conservation of Energy.

Let $\vec{r}(t)$ be the trajectory of a particle of mass m. We define the "kinetic energy" at time t by

$$KE(t) = \frac{1}{2} m \|\vec{r}\,'(t)\|^2$$

Note that we can also write

$$KE(t) = \frac{1}{2} m \left(\vec{r}\,'(t) \cdot \vec{r}\,'(t) \right),$$

so the time derivative of KE is

$$KE'(t) = \frac{m}{2} \frac{d}{dt} \left(\vec{r}\,'(t) \cdot \vec{r}\,'(t) \right)$$

product rule

$$= \frac{m}{2} \left(\vec{r}\,'(t) \cdot \vec{r}\,''(t) + \vec{r}\,''(t) \cdot \vec{r}\,'(t) \right)$$

$$= m \left(\vec{r}\,''(t) \cdot \vec{r}\,'(t) \right)$$

"mass (acceleration • velocity)"

Now let $\vec{F}$ be any force field acting on the particle. By Newton's 2nd Law:

$$\vec{F}(\vec{r}(t)) = m\,\vec{r}\,''(t).$$

Thus the total amount of work done by $\vec{F}$ on the particle between times $t = a$ & $t = b$ is

$$\int \text{work} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b (m\vec{r}\,''(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b m(\vec{r}\,''(t) \cdot \vec{r}\,'(t))\,dt$$

$$= \int_a^b KE'(t)\,dt$$

Calc I

$$= KE(b) - KE(a)$$

$$= \text{total increase in } KE.$$

On the other hand, if $\vec{F} = -\nabla f$ is a conservative force then we can also define the potential energy at time t by

$$PE(t) = f(\vec{r}(t)).$$

Then the fundamental theorem of line integrals tells us that

$$\int \vec{F}(\vec{r}(t)) \circ \vec{r}\,'(t)\, dt$$

$$= -\int \nabla f(\vec{r}(t)) \circ \vec{r}\,'(t)\, dt$$

multi-variable chain rule

$$= -\int_a^b (f \circ \vec{r})'(t)\, dt$$

Calc I

$$= -\left[(f \circ \vec{r})(b) - (f \circ \vec{r})(a) \right]$$

$$= -\left[f(\vec{r}(b)) - f(\vec{r}(a)) \right]$$

$$= -\left[PE(b) - PE(a) \right]$$

= total dec<u>rease</u> in PE

Putting the two results together shows us that

$$KE(b) - KE(a) = -[PE(b) - PE(a)]$$

$$KE(b) + PE(b) = KE(a) + PE(a)$$

We conclude that the total mechanical energy

$$E = KE + PE$$

is "conserved".

5. Gravity in Two Dimensions.

Consider the vector field

$$\vec{F} = \langle P, Q \rangle = \left\langle \frac{-x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right\rangle.$$

Note that $\vec{F}(0,0)$ is not defined.

a) If $(x,y) \neq (0,0)$ then we have

$$Q_x = \partial_x \left(-y(x^2+y^2)^{-1}\right)$$

$$= -y(-1)(x^2+y^2)^{-2}(2x)$$

$$= 2xy\,(x^2+y^2)^{-2}$$

and

$$P_y = \partial_y\left(-x\,(x^2+y^2)^{-1}\right)$$

$$= -x\,(-1)\,(x^2+y^2)^{-2}\,(2y)$$

$$= 2xy\,(x^2+y^2)^{-2},$$

so that

$$\mathrm{curl}(\vec{F})(x,y) = Q_x - P_y = 0.$$

It follows from Green's Theorem that for any loop C not containg the origin we have

$$\oint_C \vec{F}\circ\vec{T}\,ds = \iint_{\substack{\text{inside}\\ \text{the loop}}} \mathrm{curl}(\vec{F})\,dA$$

$$= \iint 0\,dA = 0.$$

b) If C_1 & C_2 are any two loops that travel once, counterclockwise, around the origin then Green's Theorem tells us that

$$\oint_{C_1} \vec{F} \cdot \vec{T}\, ds = \oint_{C_2} \vec{F} \cdot \vec{T}\, ds$$

[see the argument in the lecture.]

To compute this common value we will use our favorite curve C:

$$\vec{r}(t) = \langle \cos t, \sin t \rangle$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$$

$$\vec{F}(\vec{r}(t)) = \frac{-1}{x^2+y^2} \langle x, y \rangle$$

$$= \frac{-1}{\cos^2 t + \sin^2 t} \langle \cos t, \sin t \rangle$$

$$= \langle \cos t, \sin t \rangle .$$

The circulation around this curve is

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{2\pi} \langle \cos t, \sin t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt$$

$$= \int_0^{2\pi} (-\cos t \sin t + \sin t \cos t)\, dt$$

$$= \int_0^{2\pi} 0\, dt = 0 \qquad \text{(surprise!)}$$

Remark: Combining (a) & (b) tells us that $\oint \vec{F} \cdot \vec{T}\, ds = 0$ for <u>any</u> loop, so $\vec{F}$ is a conservative field.

Can you find the anti-derivative?

Answer:

$$f(x,y) = -\frac{1}{2} \cdot \ln(x^2+y^2).$$

Check :

$$f_x(x,y) = -\frac{1}{2} \cdot \frac{1}{x^2+y^2} \cdot (2x)$$

$$= -x/(x^2+y^2) \quad \checkmark$$

$$f_y(x,y) = \cdots = -y/(x^2+y^2) \quad \checkmark$$

How did I find $f(x,y)$?

We say that $\vec{F}$ is a <u>central</u> field if

$$\vec{F}(\vec{x}) = F(\|\vec{x}\|^2)\,\vec{x}$$

for some scalar function $F : \mathbb{R} \to \mathbb{R}$.

In this case one can check that

$$\vec{F} = \nabla\left(\tfrac{1}{2} f\right)$$

where f is any antiderivative of F.

Example :

$$F(\varphi) = \frac{-1}{\varphi} \longrightarrow \frac{1}{2} f(\varphi) = -\frac{1}{2}\ln(\varphi).$$

c) To compute the divergence:

If $(x,y) \neq (0,0)$ then we have

$$P_x = \partial_x \left(\frac{-x}{x^2+y^2} \right)$$

$$= \frac{(x^2+y^2)(-1) - (-x)(2x)}{(x^2+y^2)^2}$$

$$= (y^2 - x^2) / (x^2+y^2)^2$$

and

$$Q_y = \partial_y \left(\frac{-y}{x^2+y^2} \right)$$

$$= \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2}$$

$$= (x^2 - y^2) / (x^2+y^2)^2$$

so that

$$\operatorname{div}(\vec{F})(x,y) = P_x + Q_y = 0 .$$

It follows from the flux form of Green's Theorem that the total flux across any loop that doesn't contain the origin is zero:

$$\oint_C \vec{F} \circ \vec{N}\, ds = \iint_{\substack{\text{inside} \\ \text{the loop}}} \operatorname{div}(\vec{F})\, ds$$

$$= \iint 0 \, dA = 0$$

d) As in part (b), Green's Theorem implies that for any two loops C_1 & C_2 that each travel once, counterclockwise around the origin we must have

$$\oint_{C_1} \vec{F} \cdot \vec{N}\, ds = \oint_{C_2} \vec{F} \cdot \vec{N}\, ds$$

[Proof : Let D be the region between the curves so that $\partial D = C_1 - C_2$. Then

$$\begin{aligned} 0 &= \iint_D \mathrm{div}(\vec{F})\, ds \\ &= \oint_{C_1 - C_2} \vec{F} \cdot \vec{N}\, ds \\ &= \oint_{C_1} \vec{F} \cdot \vec{N}\, ds - \oint_{C_2} \vec{F} \cdot \vec{N}\, ds. \end{aligned}$$]

Thus we only need to compute this flux for our favorite curve C:

$$\vec{r}(t) = \langle \cos t, \sin t \rangle$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$$

$$\vec{N}(t)\, ds = \langle \cos t, -(-\sin t) \rangle\, dt.$$

$$\vec{F}(\vec{r}(t)) = \langle -\cos t, -\sin t \rangle,$$

so that

$$\oint_C \vec{F} \cdot \vec{N}\, ds$$

$$\vec{N}(t)\,dt$$

$$= \int \vec{F}(\vec{r}(t)) \cdot \langle \cos t, \sin t \rangle\, dt$$

$$= \int \langle -\cos t, -\sin t \rangle \cdot \langle \cos t, \sin t \rangle\, dt$$

$$= \int (-\cos^2 t - \sin^2 t)\, dt$$

$$= \int_0^{2\pi} (-1)\, dt = -2\pi.$$

For any simple counterclockwise loop C we conclude that

$$\oint_C \vec{F} \cdot \vec{N}\, ds = \begin{cases} 0 & \text{if } C \text{ does not contain } (0,0) \\ -2\pi & \text{if } C \text{ contains } (0,0) \end{cases}$$

Tomorrow we'll discuss what this has to do with gravity & electric forces!