**Math 211** | **Summer 2021**
**Homework 4** | **Drew Armstrong**

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**Problem 1. Practice with the Chain Rule.** Let $f(x,y)$ be a function of $x$ and $y$, where $x = s^2 + t^2$ and $y = 2st$ are functions of $s$ and $t$.

(a) Express $f_s$ and $f_t$ in terms of $s$, $t$, $f_x$ and $f_y$.
(b) Express $f_{ss}$ in terms of $s$, $t$, $f_{xx}$, $f_{xy}$ $(= f_{yx})$ and $f_{yy}$. [Hint: As an intermediate step you will need to compute $f_{xs}$ and $f_{ys}$. To do this, think of $f_x(x,y)$ and $f_y(x,y)$ as functions of $(x,y)$ and use the chain rule as in part (a).]

**Problem 2. Integration over a Rectangle.** Find the volume between the surface $z = (x+y)^2$ and a general rectangle in the $xy$-plane: $a_1 \leq x \leq a_2$ and $b_1 \leq y \leq b_2$.

**Problem 3. Integration over a Tetrahedron.** Consider the solid tetrahedron with $x, y, z \geq 0$ and $x + 2y + 3z \leq 6$. Suppose that this solid has a mass density of $\rho(x,y,z) = 1 + x$.

(a) Compute the total mass $m = \iiint \rho(x,y,z)\, dV$.
(b) Compute the moments about the three coordinate planes:

$$M_{yz} = \iiint x\rho\, dV,$$
$$M_{xz} = \iiint y\rho\, dV,$$
$$M_{xy} = \iiint z\rho\, dV.$$

(c) Find the center of mass of the solid tetrahedron.

[Hint: See the note on "parametrizing a tetrahedron".]

**Problem 4. Cylindrical Coordinates.** Consider the parabolic dome between the unit circle $x^2 + y^2 \leq 1$ and the surface $z = 1 - x^2 - y^2$. We will compute the center of mass, assuming that the density is constant: $\rho(x,y,z) = 1$.

(a) Use cylindrical coordinates to compute the mass $m = \iiint 1\, dV$. [Hint: Integrate over $z$ first. We already did all of the work in class.]
(b) Use cylindrical coordinates to compute the $xy$-moment $M_{xy} = \iiint z\, dV$.
(c) Find the center of mass. [Hint: Because the shape has rotational symmetry around the $z$-axis you can assume that $M_{xz} = M_{yz} = 0$.]

**Problem 5. Spherical Coordinates.** Consider the solid region in $\mathbb{R}^3$ above the $xy$-plane, below the cone $z = \sqrt{x^2 + y^2}$ and inside the unit sphere $x^2 + y^2 + z^2 \leq 1$.

(a) Use spherical coordinates to compute the volume of this region. [Hint: The region can be parametrized with $\rho$ from 0 and 1 and $\theta$ from 0 to $2\pi$. What about $\varphi$?]
(b) Compute the moment $M_{xy} = \iiint z dV$ and the center of mass. [Hint: Because the shape has rotational symmetry around the $z$-axis you can assume that $M_{xz} = M_{yz} = 0$.]