1. Eliminate the parameter and draw the parametrized curve.

(a) $(x,y) = (6-4t, -1+3t)$

$$x = 6-4t \rightsquigarrow t = \frac{x-6}{-4}$$

$$y = -1+3t \rightsquigarrow t = \frac{y+1}{3}$$

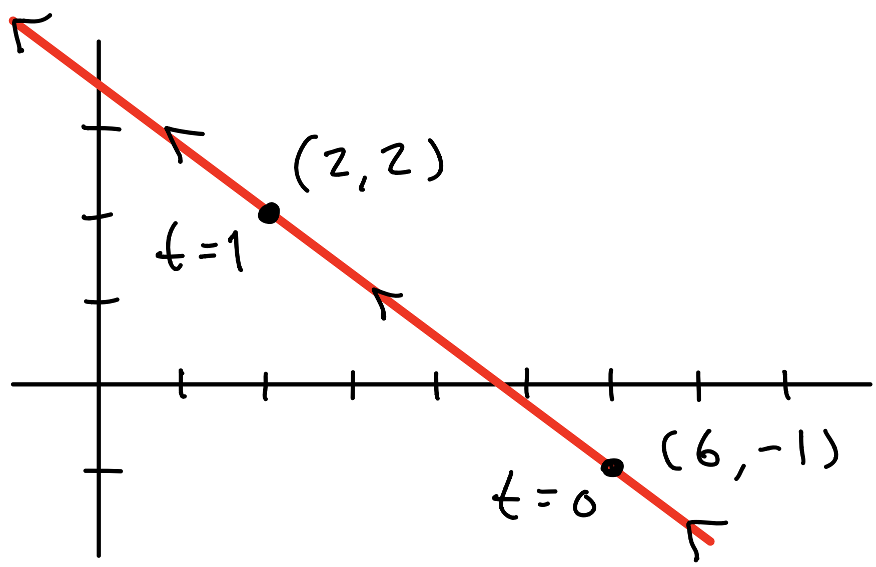
so $$\frac{y+1}{3} = \frac{x-6}{-4}$$

$$-4y-4 = 3x-18$$

$$-4y = 3x-14$$

$$y = -\frac{3}{4}x + \frac{7}{2}$$

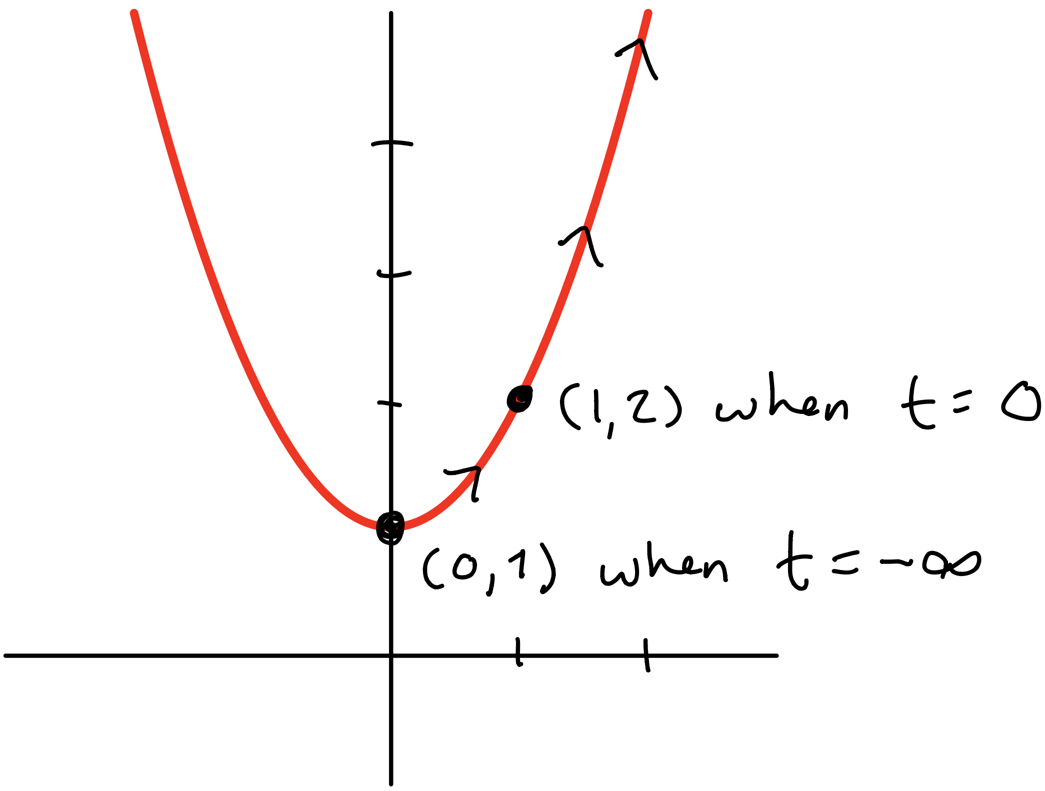
Picture:



(b) $(x, y) = (e^t, e^{2t} + 1)$

$$x = e^t$$
$$x^2 = (e^t)^2 = e^{2t}$$
$$x^2 + 1 = e^{2t} + 1 = y$$
$$y = x^2 + 1$$

Picture:



Actually we only get the right half of the parabola because $e^t > 0$ for all real values of t.

(c) $(x,y) = (1+\cos t,\ 2+\sin t)$

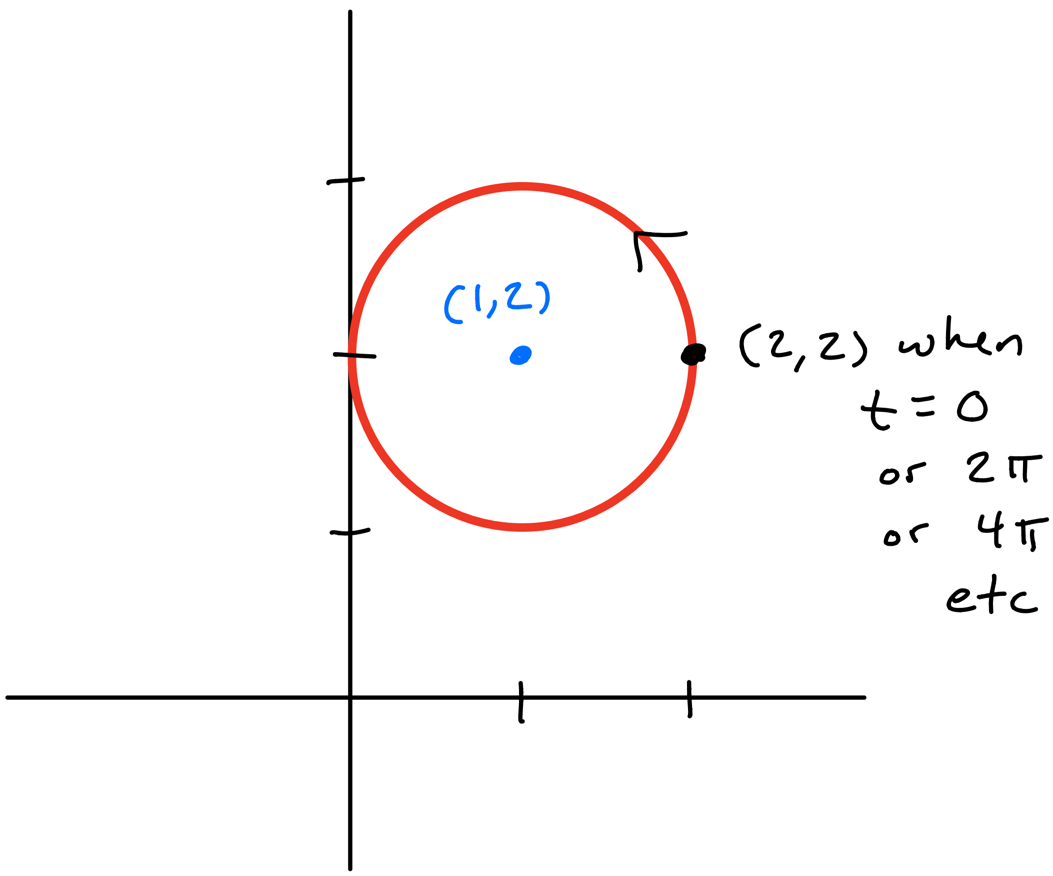
$x = 1+\cos t$ $\qquad$ $y = 2+\sin t$

$x-1 = \cos t$ $\qquad$ $y-2 = \sin t$

So $(x-1)^2+(y-2)^2 = \cos^2 t+\sin^2 t$

$(x-1)^2+(y-2)^2 = 1.$

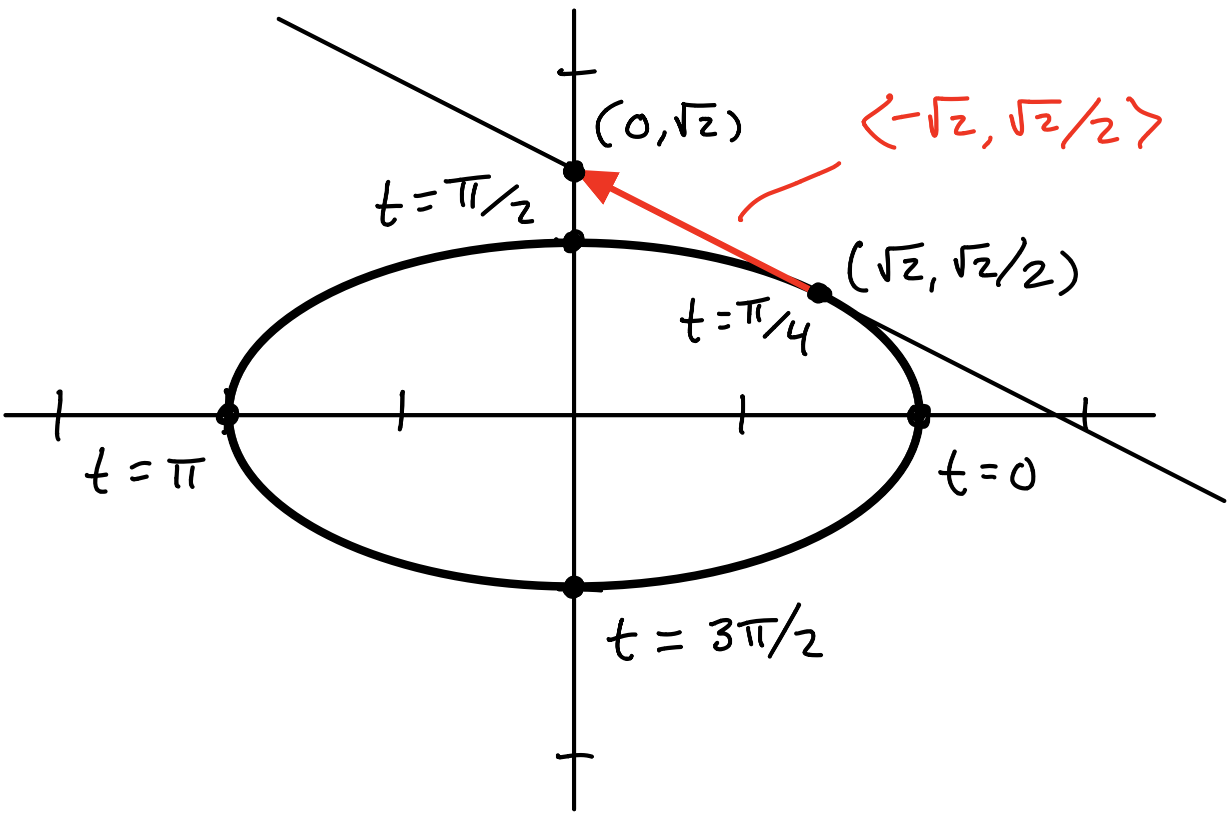
Picture :



2. A parametrized ellipse

$$\left(\frac{x}{2}\right)^2 + y^2 = 1$$

$$(x, y) = (2\cos t, \sin t)$$



velocity at time t:

$$\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \langle -2\sin t, \cos t \rangle$$

speed at time t:

$$\sqrt{(-2\sin t)^2 + \cos^2 t} = \sqrt{4\sin^2 t + \cos^2 t}$$

At time $t = \pi/4$ we have

position $= (\sqrt{2}, \sqrt{2}/2)$

velocity $= \langle -\sqrt{2}, \sqrt{2}/2 \rangle$

The slope of the tangent line at time t is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-2\sin t}$$

and at $t = \pi/4$ is

$$\frac{\cos(\pi/4)}{-2\sin(\pi/4)} = \frac{\sqrt{2}/2}{-\sqrt{2}} = -\frac{1}{2}$$

Use point-slope to get the equation of the tangent line at $t = \pi/4$:

$$(y - y_0) = m(x - x_0)$$

$$y - \sqrt{2}/2 = -\frac{1}{2}(x - \sqrt{2})$$

$$y = -\frac{1}{2}x + \sqrt{2}$$

[There are several different ways to get this equation.]

The perimeter is the arc length between $t = 0$ and $t = 2\pi$:

$$\text{perimeter} = \int \text{speed}\, dt$$

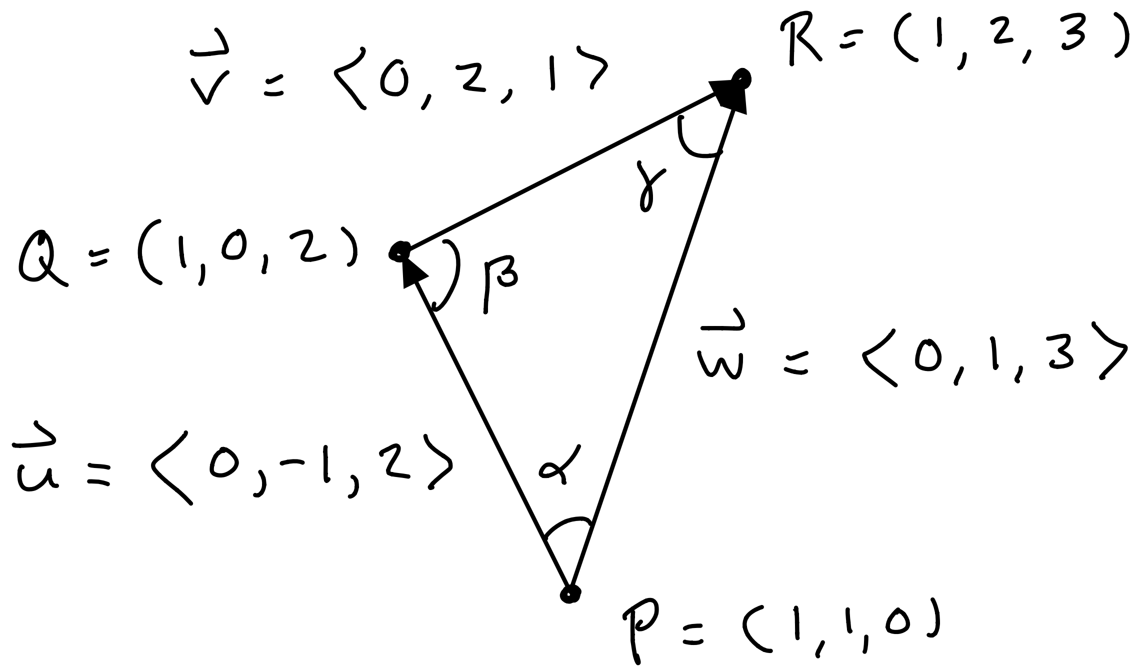
$$= \int_0^{2\pi} \sqrt{4\sin^2 t + \cos^2 t}\, dt$$

$$\approx 9.69$$

This integral cannot be solved by elementary means (the same is true for most arc lengths).

3. A triangle in space.

$P = (1,1,0)$, $Q = (1,0,2)$, $R = (1,2,3)$



[Why did I choose to draw it like this?]

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{0^2 + (-1)^2 + 2^2} = \sqrt{5}$$

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{0^2 + 2^2 + 1^2} = \sqrt{5}$$

$$\|\vec{w}\| = \sqrt{\vec{w} \cdot \vec{w}} = \sqrt{0^2 + 1^2 + 3^2} = \sqrt{10}$$

$$\vec{u} \cdot \vec{w} = \|\vec{u}\| \, \|\vec{w}\| \cos \alpha$$

$$0 \cdot 0 + (-1) 1 + 2 \cdot 3 = \sqrt{5} \sqrt{10} \cos \alpha$$

$$\cos \alpha = 5 / \sqrt{5}\sqrt{10} = 1/\sqrt{2}$$

$\rightsquigarrow \alpha = \pi/4 = 45^\circ$ exactly!

$$(-\vec{u}) \circ \vec{v} = \|-\vec{u}\| \|\vec{v}\| \cos \beta$$

$$0 \cdot 0 + 1 \cdot 2 + (-2) 1 = \sqrt{5} \sqrt{5} \cos \beta$$

$$\cos \beta = 0$$

$$\rightsquigarrow \quad \beta = 90^\circ$$

$[\ \vec{u} \circ \vec{v} = 0$ so $\vec{u} \perp \vec{v} = 0\ ]$

$$(-\vec{v}) \circ (-\vec{w}) = \|-\vec{v}\| \|-\vec{w}\| \cos \gamma$$

$$0 \cdot 0 + (-2)(-1) + (-1)(-3) = \sqrt{5} \sqrt{10} \cos \gamma$$

$$\cos \gamma = 5 / \sqrt{5}\sqrt{10} = 1/\sqrt{2}$$

$$\rightsquigarrow \gamma = \pi/4 = 45^\circ$$

[Actually we only needed to compute two of the angles because $\alpha + \beta + \gamma = \pi = 180^\circ$.]

4. Vector Arithmetic.

$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$

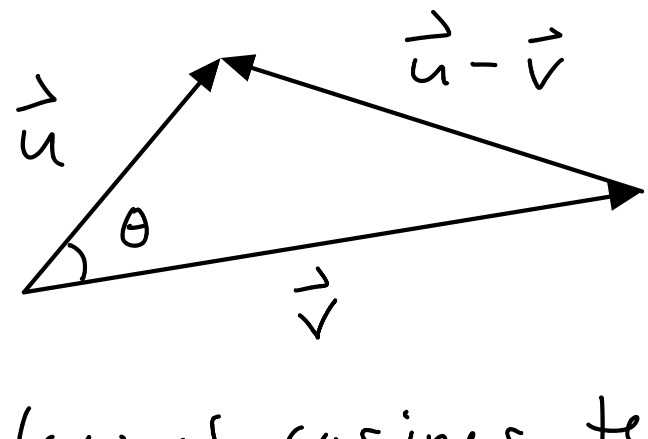
$$= \vec{u}\cdot(\vec{u}-\vec{v}) - \vec{v}\cdot(\vec{u}-\vec{v})$$

$$= \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$

$$= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - 2\vec{u}\cdot\vec{v}$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u}\cdot\vec{v}$$

We can think of this as a relationship between the side lengths of a triangle of vectors:



Then the law of cosines tells us

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta.$$

5. Equations of Lines & Planes

(a) The line passing through $(-1,2)$ & perpendicular to $\langle 3,1\rangle$ has the equation

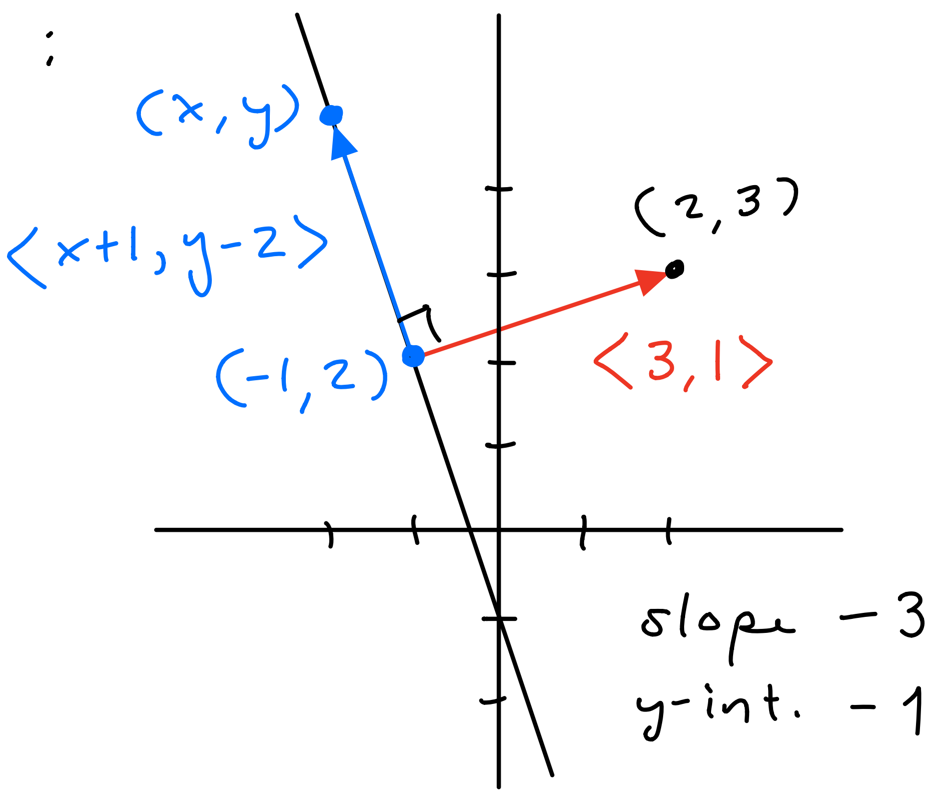
$$3(x-(-1)) + 1(y-2) = 0$$

$$y-2 = -3(x+1)$$

$$y = -3x - 3 + 2$$

$$y = -3x - 1$$

Picture :



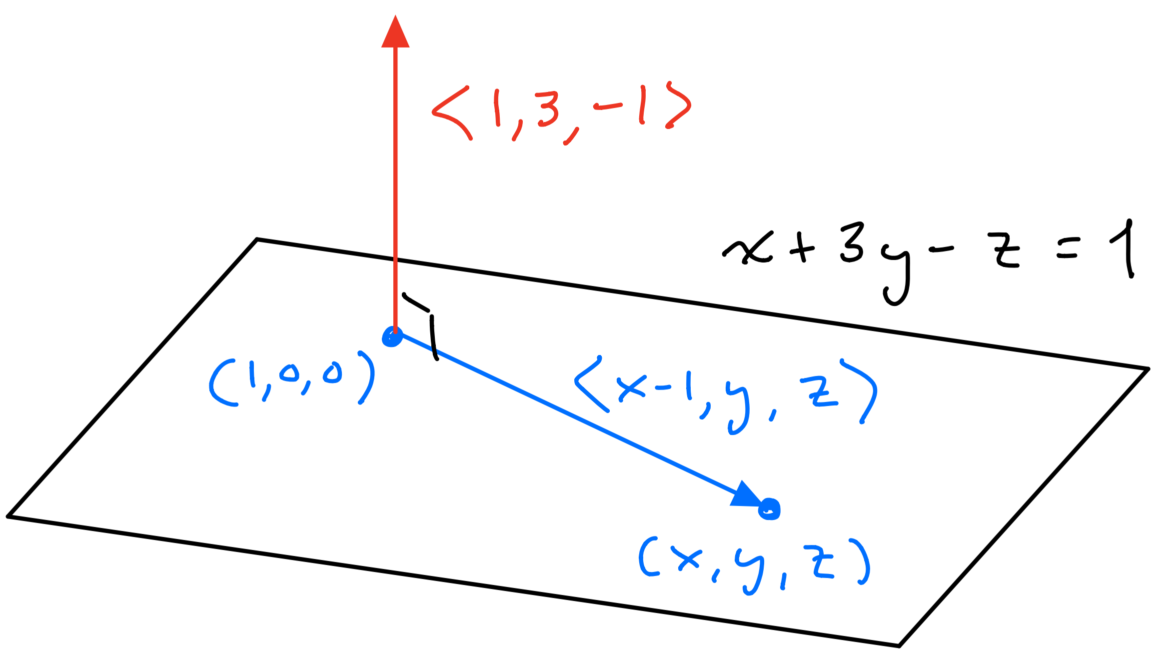
(b) The plane passing through point $(1,0,0)$ and perpendicular to the vector $\langle 1,3,-1\rangle$ has the equation

$$1(x-1)+3(y-0)+(-1)(z-0)=0$$

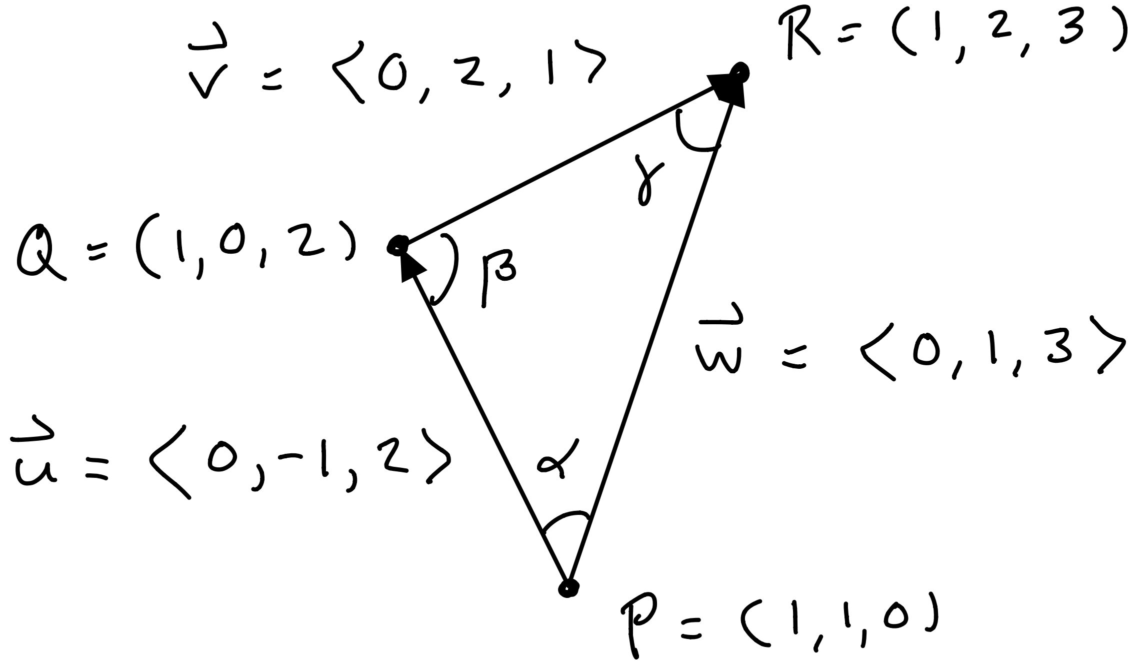
$$x-1+3y-z=0$$

$$x+3y-z=1$$

Picture:



(c) Find the equation of the plane containing the triangle from Problem 3



We have plenty of points in the plane but we need a normal vector. TRICK: Take the cross product of <u>any</u> two vectors in the plane, e.g.,

$$\vec{n} = \vec{u} \times \vec{v}$$

$$= \langle 0, -1, 2 \rangle \times \langle 0, 2, 1 \rangle$$

$$= \langle -1\cdot 1 - 2\cdot 2 \, , \, 2\cdot 0 - 0\cdot 1 \, , \, 0\cdot 2 - (-1)\cdot 0 \rangle$$

$$= \langle -5, 0, 0 \rangle$$

Taking $(x_0, y_0, z_0) = (1, 1, 0)$ and $\langle a, b, c \rangle = \langle -5, 0, 0 \rangle$, we obtain the equation of the plane:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$-5(x-1) + 0(y-1) + 0(z-0) = 0$$

$$-5x + 5 = 0$$

$$-5x = -5$$

$$x = 1.$$

wow, what a simple equation!
We can think of this as a copy of the (y,z)-plane because the x coordinates are always the same. [That's how I drew the picture.]

Preview: There is another way to describe this plane using 2 "parameters", because the plane is "2 dimensional".

It turns out that any point of the plane containing a triangle

P, Q, R can be expressed in the form

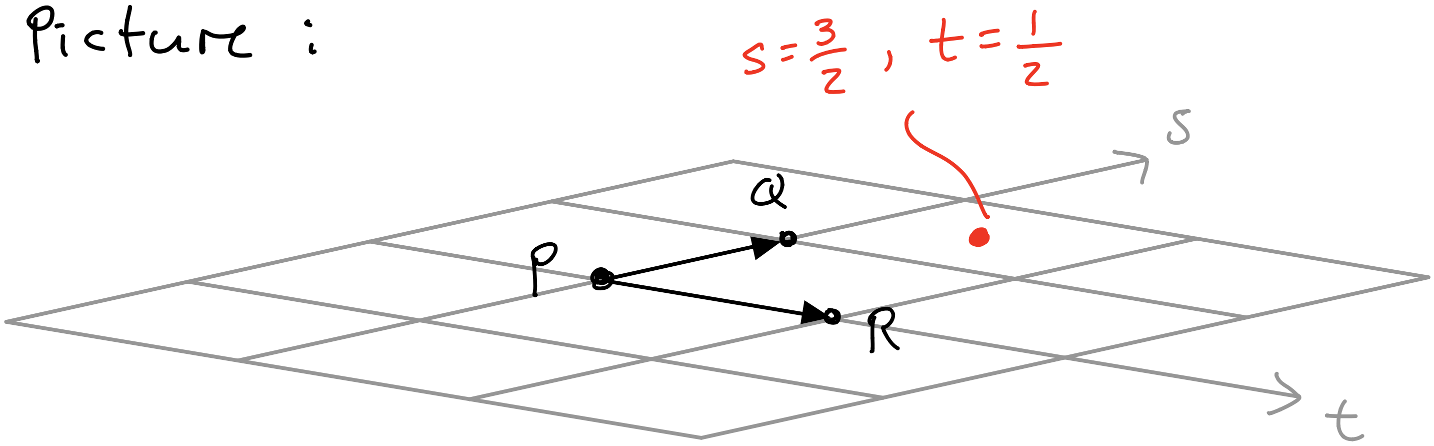
$$(x,y,z) = P + s\,\overrightarrow{PQ} + t\,\overrightarrow{PR}$$

In our case,

$$(x,y,z) = (1,1,0) + s\langle 0,-1,2\rangle + t\langle 0,1,3\rangle$$

$$\boxed{(x,y,z) = (1,\, 1-s+t,\, 2s+3t)}$$

Picture :



Every point in the plane has unique "(s,t)-coordinates".