Math 211 Exam 2 | Fall 2019 Drew Armstrong

There are 6 pages, each worth 6 points, for a total of 36 points. This is a closed book test. No electronic devices are allowed. Show your work for full credit.

Problem 1. Consider the function $f(x, y, z) = xyz^2$.

(a) Compute the gradient vector ∇f at the point $(3, 2, 1)$.

$$\nabla f = \langle yz^2, xz^2, 2xy \rangle$$

$$\nabla f(3,2,1) = \langle 2, 3, 12 \rangle$$

(b) Find the equation of the tangent plane to the surface $f(x, y, z) = 6$ at the point $(3, 2, 1)$.

$$\nabla f(3,2,1) \cdot (\vec{x} - \langle 3,2,1 \rangle) = 0$$

$$2(x-3) + 3(y-2) + 12(z-1) = 0$$

(c) Compute the rate of change of f at the point $(3, 2, 1)$ in the direction of $\boxed{\overset{\vec{u}}{\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle}}$.

$$D_{\vec{u}} f(3,2,1) = \nabla f(3,2,1) \cdot \vec{u}$$

$$= \langle 2, 3, 12 \rangle \cdot \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

$$= \frac{17}{\sqrt{3}}$$

Problem 2. Consider the function $f(x, y) = x^2y - x^2 - y^2$.

(a) This function has three critical points. Find them.

$f_x = 2xy - 2x = 2x(y-1) = 0$

$\Longrightarrow \quad x = 0 \quad \text{or} \quad y = 1.$

$f_y = x^2 - 2y = 0. \Longrightarrow \quad x^2 = 2y.$

If $x = 0$ then $y = 0$.

If $y = 1$ then $x = \pm\sqrt{2}$.

Critical Points: $(0,0)$, $(+\sqrt{2}, 1)$, $(-\sqrt{2}, 1)$

(b) Use the second derivative test to determine whether each critical point is a local maximum, a local minimum, or a saddle point.

$$\left.\begin{aligned} f_{xx} &= 2(y-1) \\ f_{yy} &= -2 \\ f_{xy} &= f_{yx} = 2x \end{aligned}\right\} \quad D = -4(y-1) - 4x^2.$$

$(0,0) \Longrightarrow D = 4 > 0 \ \& \ f_{xx} = -2 < 0$

LOCAL MAX

$(\pm\sqrt{2}, 1) \Longrightarrow D = -8 < 0$

SADDLE POINTS.

Problem 3. We will use Lagrange multipliers to optimize the function $f(x,y,z) = x + y + z$ subject to the constraint $g(x,y,z) = x^2 + 2y^2 + 3z^2 = 1$.

(a) Use the vector equation $\nabla f = \lambda \nabla g$ to show that $(x,y,z) = (x, x/2, x/3)$.

$$\langle 1,1,1\rangle = \lambda \langle 2x, 4y, 6z\rangle$$

$$\left\{\begin{array}{l} 1 = \lambda 2x \\ 1 = \lambda 4y \\ 1 = \lambda 6y \end{array}\right\} \Longrightarrow \begin{array}{l} \cancel{\lambda} 2x = \cancel{\lambda} 4y = \cancel{\lambda} 6z \\ x = 2y = 3z \end{array}$$

$$\Longrightarrow y = \frac{x}{2} \quad \& \quad z = \frac{x}{3}.$$

(b) Substitute this into the equation $g(x,y,z) = 1$ to obtain values for (x,y,z).

$$x^2 + 2y^2 + 3z^2 = 1$$

$$x^2 + 2\frac{x^2}{4} + 3\frac{x^2}{9} = 1$$

$$x^2 + \frac{x^2}{2} + \frac{x^2}{3} = 1$$

$$6x^2 + 3x^2 + 2x^2 = 6$$

$$11x^2 = 6$$

$$x = \pm\sqrt{6/11}, \quad y = \pm\frac{1}{2}\sqrt{6/11}, \quad z = \pm\frac{1}{3}\sqrt{6/11}.$$

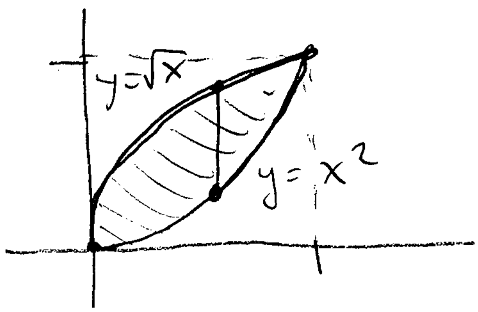
(c) Finally, compute the maximum and minimum values of f assuming that $g = 1$.

MAX: $f = \sqrt{\frac{6}{11}} + \frac{1}{2}\sqrt{\frac{6}{11}} + \frac{1}{3}\sqrt{\frac{6}{11}}$

MIN: $f = -\sqrt{\frac{6}{11}} - \frac{1}{2}\sqrt{\frac{6}{11}} - \frac{1}{3}\sqrt{\frac{6}{11}}$

Problem 4. Let D be the region between the curves $y = x^2$ and $y = \sqrt{x}$, where $0 \leq x \leq 1$.

(a) Draw a picture of the region.



(b) Write down a parametrization for this region.

$$0 \leq x \leq 1$$
$$x^2 \leq y \leq \sqrt{x}$$

(c) Use your parametrization from part (b) to compute the integral $\iint_D y\, dA$.

$$\int_0^1 \int_{x^2}^{\sqrt{x}} y\, dy\, dx = \int_0^1 \left[\frac{1}{2} y^2\right]_{x^2}^{\sqrt{x}} dx$$

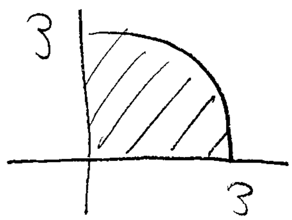
$$= \int_0^1 \frac{1}{2}\left(x - x^4\right) dx = \left[\frac{1}{2}\left(\frac{x^2}{2} - \frac{x^5}{5}\right)\right]_0^1$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{1}{2}\,\frac{3}{10} = \frac{3}{20}$$

Problem 5. Consider the following double integral in Cartesian coordinates:

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx.$$

(a) Rewrite this integral in polar coordinates.



$0 \leq r \leq 3$

$0 \leq \theta \leq \frac{\pi}{2}$

$$\int_0^{\pi/2} \int_0^{3} \frac{1}{r} \cdot r\, dr\, d\theta$$

(b) Use your answer from part (a) to compute the integral.

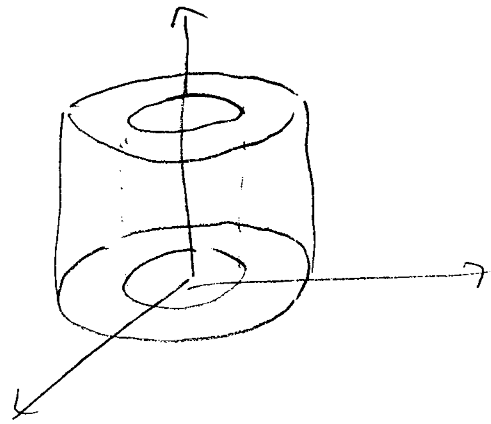
$$\int_0^{\pi/2} \int_0^{3} 1\, dr\, d\theta$$

$$\int_0^{\pi/2} 3\, d\theta = 3 \cdot \frac{\pi}{2}.$$

Problem 6. Let E be the solid region between the cylinders $x^2+y^2=1$ and $x^2+y^2=4$, and between the planes $z=0$ and $z=1$.

(a) Write down a parametrization for this region using cylindrical coordinates.

$$1 \leq r \leq 2$$
$$0 \leq \theta \leq 2\pi$$
$$0 \leq z \leq 1$$



(b) Use your parametrization from part (a) to compute the integral $\iiint_E z\,dV$.

$$\int_0^1 \int_0^{2\pi} \int_1^2 z \underbrace{r\,dr\,d\theta\,dz}_{dV}$$

$$= \int_0^1 \int_0^{2\pi} z \frac{r^2}{2}\Big|_1^2 \,d\theta\,dz = \int_0^1 \int_0^{2\pi} z \cdot \frac{3}{2}\,d\theta\,dz$$

$$= \int_0^1 z \cdot \frac{3}{2} \cdot 2\pi\,dz = 3\pi \frac{z^2}{2}\Big|_0^1 = \frac{3\pi}{2}$$