Math 211
Exam 2 Sample

Fall 2019
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There are 6 pages, each worth 6 points, for a total of 36 points. This is a closed book test. No electronic devices are allowed. Show your work for full credit.

1. Based on Exercise 11.6.33. Consider the function $f(x, y) = ye^{-x}$.

(a) Find a unit vector $\mathbf{u}$ so that the rate of change of $D_{\mathbf{u}} f$ is maximized at the point $(0, 0)$.

$\nabla f = \langle -ye^{-x}, e^{-x} \rangle$

$\nabla f(0,0) = \langle 0, 1 \rangle$

$\vec{u} = \langle 0, 1 \rangle$

(b) Find a unit vector $\mathbf{u}$ so that the rate of change $D_{\mathbf{u}} f$ is maximized at the point $(0, 1)$.

$\nabla f(0,1) = \langle -1, 1 \rangle$

$\vec{u} = \frac{1}{\sqrt{2}} \langle -1, 1 \rangle$

(c) Find the rate of change of f at the point $(0, 1)$ in the direction of $\mathbf{v} = \langle \sqrt{3}/2, -1/2 \rangle$.

Since $\vec{v}$ is a unit vector,

$D_{\vec{v}} f(0,1) = \nabla f(0,1) \cdot \vec{v}$

$= \langle -1, 1 \rangle \cdot \langle \sqrt{3}/2, -1/2 \rangle$

$= \frac{-\sqrt{3}}{2} - \frac{1}{2} = \frac{-\sqrt{3}-1}{2}$

2. Based on Exercise 11.7.15. Consider the function

$$f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2.$$

Find the critical points and use the second derivative test to classify them local maximal, local minima, or saddle points.

$$\begin{cases} ① \; f_x = 6xy - 6x = 0 \rightsquigarrow 6x(y-1) = 0 \\ ② \; f_y = 3x^2 + 3y^2 - 6y = 0 \end{cases}$$

① Says $x = 0$ or $y = 1$.

IF $x = 0$ then ② says $3y(y-2) = 0$

so $y = 0$ or $y = 2$

If $y = 1$ then ② says $x = \pm 1$.

$f_{xx} = 6y - 6 \quad f_{xy} = 6x \quad D = f_{xx} f_{zz} - f_{xy}^2$

$f_{yy} = 6y - 6 \quad f_{yx} = 6x \quad = (6y-6)^2 - (6x)^2$

$D(0,0) = 36 > 0$ and $f_{xx}(0,0) = -6 < 0$

$\Rightarrow (0,0)$ is local MAX

$D(0,2) = 36 > 0$ and $f_{xx}(0,2) = +6 > 0$

$\Rightarrow (0,2)$ Local MIN.

$$\left.\begin{aligned} D(+1,1) &= -36 < 0 \\ D(-1,1) &= -36 < 0 \end{aligned}\right\} \Rightarrow (\pm 1, 1) \text{ are SADDLE points.}$$

3. Based on Exercise 11.8.7. We will use the method of Lagrange multipliers to maximize the volume of a box $V(x,y,z) = xyz$ subject to the constraint $g(x,y,z) = x^2 + 2y^2 + 3z^2 = 6$.

(a) Use the vector equation $\nabla V = \lambda \nabla g$ to show that $(x,y,z) = (x, \pm x/\sqrt{2}, \pm x/\sqrt{3})$.

$$\left.\begin{array}{l}\nabla V = \langle yz, xz, xy\rangle \\ \nabla g = \langle 2x, 4y, 6z\rangle\end{array}\right\} \Longrightarrow \begin{array}{l} yz = \lambda 2x \\ xz = \lambda 4y \\ xy = \lambda 6z \end{array}$$

$$\Longrightarrow \lambda = \frac{yz}{2x} = \frac{xz}{4y} = \frac{xy}{6z}.$$

$$\Longrightarrow 4y^2z = 2x^2z \quad \& \quad 6yz^2 = 2x^2y$$

$$y = \pm x/\sqrt{2} \qquad z = \pm x/\sqrt{3}.$$

(b) Substitute this into the constraint equation $g(x,y,z) = 6$ to obtain values for (x,y,z).

$$g(x,y,z) = 6$$

$$g\left(x, \pm\frac{x}{\sqrt{2}}, \pm\frac{x}{\sqrt{3}}\right) = 6$$

$$x^2 + x^2 + x^2 = 6$$

$$3x^2 = 6$$

$$x = \pm\sqrt{2}, \quad y = \pm 1, \quad z = \pm\sqrt{2}/\sqrt{3}.$$

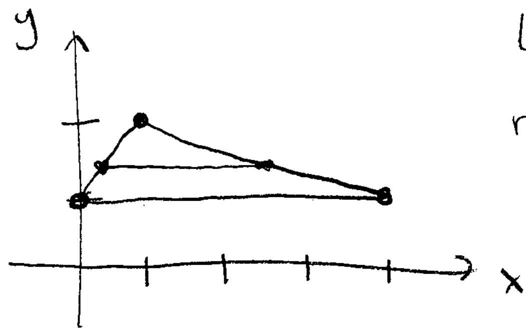
(c) Plug in these values into $V(x,y,z)$ to obtain the maximum volume.

$$V = (\pm\sqrt{2})(\pm 1)\left(\pm\frac{\sqrt{2}}{\sqrt{3}}\right) = \pm 2/\sqrt{3}.$$

Max volume is $+2/\sqrt{3}$.

4. Based on Exercise 12.2.17. Let D be the triangular region with vertices $(0,1)$, $(1,2)$, $(4,1)$.

(a) Draw a picture of the region.



left Line : $y = x + 1$

right Line : $y = -\frac{x}{3} + \frac{7}{3}$

(b) Write down a parametrization for this region.

$$1 \leq y \leq 2$$

$$y - 1 \leq x \leq 7 - 3y$$

(c) Use your parametrization from part (b) to compute the integral $\iint_D y^2\, dA$.

$$\int_1^2 \left[\int_{y-1}^{7-3y} y^2 \, dx \right] dy = \int_1^2 y^2 \overbrace{\left[(7-3y) - (y-1) \right]}^{8-4y} dy$$

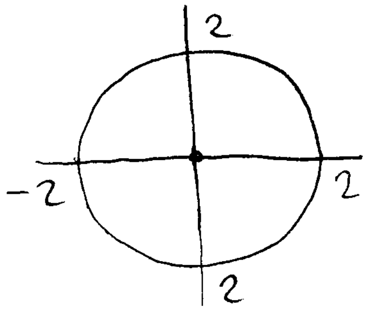
$$= \int_1^2 8y^2 - 4y^3 \, dy = \left[\frac{8}{3} y^3 - y^4 \right]_1^2$$

$$= \left(\frac{64}{3} - 16 \right) - \left(\frac{8}{3} - 1 \right) = \frac{11}{3}$$

5. Based on Exercise 12.3.13. The following integral computes the volume of the solid below the cone $z = \sqrt{x^2+y^2}$ and above the disk $x^2+y^2 \leq 4$:

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\, dy\, dx.$$

(a) Rewrite this integral in polar coordinates.



$0 \leq \theta \leq 2\pi$

$0 \leq r \leq 2$

$$\int_0^{2\pi}\int_0^{2} r \cdot r\, dr\, d\theta$$

(b) Solve your integral from part (a) to compute the volume of the solid.

$$= \int_0^{2\pi}\left[\int_0^{2} r^2\, dr\right] d\theta$$

$$= \int_0^{2\pi}\left(\frac{2^3}{3} - \frac{0^3}{3}\right) d\theta$$

$$= 2\pi \cdot \frac{8}{3}$$

6. Based on Example 12.6.4. A circular cone with radius R and height ~~h~~ R has the following parametrization in cylindrical coordinates:

$$E = \{(r, \theta, z) : 0 \le r \le R, 0 \le \theta \le 2\pi, r \le z \le \cancel{h}\, R\}.$$

Compute the volume of the cone.

$$\iiint_E dV = \int_r^R \int_0^{2\pi} \int_0^R r\,dr\,d\theta\,dz$$

$$= \int_0^{2\pi} \int_0^R \left[\int_r^R r\,dz \right] dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^R r(R-r)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{R^3}{2} - \frac{R^3}{3} \right) d\theta = 2\pi \left(\frac{1}{6} R^3 \right)$$

$$= \frac{\pi R^3}{3}$$