Math 211 **Fall 2019**
Exam 1 **Drew Armstrong**

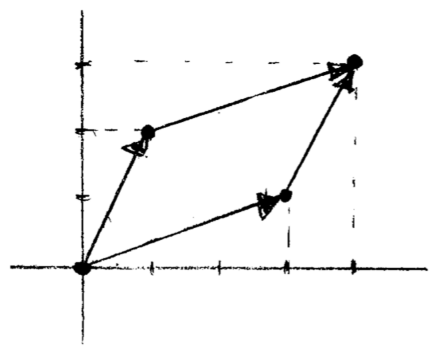
There are 6 pages, each worth 6 points, for a total of 36 points. This is a closed book test. No electronic devices are allowed. Show your work for full credit.

Problem 1. Let θ be the angle between the vectors $\mathbf{u} = \langle 3, 1\rangle$ and $\mathbf{v} = \langle 1, 2\rangle$.

(a) Compute $\cos\theta$.

$$\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{3\cdot 1 + 1\cdot 2}{\sqrt{3^2+1^2}\sqrt{1^1+2^2}} = \frac{5}{\sqrt{10}\sqrt{5}} = \frac{1}{\sqrt{2}} \qquad [\theta = 45°]$$

(b) Draw the parallelogram defined by $\mathbf{u}$ and $\mathbf{v}$.



(c) Compute the area of the parallelogram. [Hint: The formula involves $\sin\theta$. Recall the identity $\sin\theta = \sqrt{1-\cos^2\theta}$.]

Solution 1:

$$\text{area } = |\mathbf{u}||\mathbf{v}|\sin\theta = |\mathbf{u}||\mathbf{v}|\sqrt{1-\cos^2\theta} = \sqrt{10}\sqrt{5}\sqrt{1-1/2} = 5$$

Solution 2:

$$\text{area } = |\langle 3,1,0\rangle \times \langle 1,2,0\rangle| = |\langle 0,0,5\rangle| = 5$$

Solution 3:

$$\text{area } = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 3\cdot 2 - 1\cdot 1 = 5$$

2. Consider the points $P(1,1,1)$, $Q(1,2,3)$ and $R(2,1,2)$.

(a) Use the cross product to find a vector that is perpendicular to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 0,1,2\rangle \times \langle 1,0,1\rangle = \langle 1,2,-1\rangle$$

(b) Find an equation for the plane defined by P, Q, R.

$$\begin{aligned} a(x-x_0)+b(y-y_0)+c(z-z_0)&=0\\ 1(x-1)+2(y-1)-1(z-1)&=0\\ x+2y-z&=2 \end{aligned}$$

(c) Determine whether the point $S(0,2,2)$ is on this plane.

$$\text{Yes because } (0)+2(2)-(2)=2.$$

Problem 3. Two particles are traveling in the plane along the following curves:

$$\mathbf{r}_1(t)=\langle t,t^2\rangle \qquad \text{and} \qquad \mathbf{r}_2(t)=\langle 4,0\rangle+t\langle -1,2\rangle.$$

(a) I claim that the two particles will collide. Find the value of t when this happens.

$$\mathbf{r}_1(t)=\langle 2,4\rangle=\mathbf{r}_2(t) \text{ when } t=2$$

(b) Find the velocity vectors of the two particles at the moment that they collide.

$$\begin{aligned} \mathbf{r}_1'(t)&=\langle 1,2t\rangle,\\ \mathbf{r}_1'(2)&=\langle 1,4\rangle,\\ \mathbf{r}_2'(t)&=\langle -1,2\rangle,\\ \mathbf{r}_2'(2)&=\langle -1,2\rangle. \end{aligned}$$

(c) Find (the cosine of) the angle between these two velocities (i.e., the angle of collision).

$$\cos\theta=\frac{\langle 1,4\rangle\times\langle -1,2\rangle}{|\langle 1,4\rangle||\langle -1,2\rangle|}=\frac{7}{\sqrt{17}\sqrt{5}} \qquad [\theta=40.6^\circ]$$

Problem 4. The velocity of a particle at time t is given by

$$\mathbf{v}(t)=\mathbf{r}'(t)=\langle 3,4\cos t,4\sin t\rangle.$$

(a) Compute the acceleration vector $\mathbf{a}(t)=\mathbf{v}'(t)=\mathbf{r}''(t)$ at time t.

$$\mathbf{a}(t)=\langle 0,-4\sin t,4\cos t\rangle$$

(b) Supppose that the initial position is $\mathbf{r}(0) = \langle 0, 0, 0\rangle$. Compute the position $\mathbf{r}(t)$ of the particle at time t.

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{v}(t)\,dt = \left\langle \int 3\,dt, \int 4\cos t\,dt, \int 4\sin t\,dt \right\rangle \\ &= \langle 3t + c_1, 4\sin t + c_2, -4\cos t + c_3\rangle\end{aligned}$$

$$\begin{aligned}\mathbf{r}(0) &= \langle 3(0) + c_1, 4\sin(0) + c_2, -4\cos(0) + c_3\rangle \\ \langle 0, 0, 0\rangle &= \langle c_1, c_2, -4 + c_3\rangle \\ \langle 0, 0, 4\rangle &= \langle c_1, c_2, c_3\rangle\end{aligned}$$

$$\mathbf{r}(t) = \langle 3t + 0, 4\sin t + 0, -4\cos t + 4\rangle$$

(c) Compute the distance traveled by the particle between times $t = 0$ and $t = 1$.

$$\begin{aligned}\text{distance} &= \int_0^1 |\mathbf{v}(t)|\,dt \\ &= \int_0^1 \sqrt{3^2 + 4^2\cos^2 t + 4^2 \sin^2 t}\,dt \\ &= \int_0^1 \sqrt{3^2 + 4^2}\,dt \\ &= \int_0^1 \sqrt{25}\,dt \\ &= \int_0^1 5\,dt = 5.\end{aligned}$$

Problem 5. Consider the two-variable function $f(x, y) = x^2 + xy + 3y^2$.

(a) Compute the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$.

$$\begin{aligned}f_x(x, y) &= 2x + y, \\ f_y(x, y) &= x + 6y.\end{aligned}$$

(b) Find and equation for the tangent plane to the surface $z = f(x, y)$ when $(x, y) = (2, 1)$.

$$\begin{aligned}(z - f(2, 1)) &= (x - 2)f_x(2, 1) + (y - 1)f_y(2, 1) \\ (z - 9) &= 5(x - 2) + 8(y - 1) \\ z &= 9 + 5(x - 2) + 8(y - 1)\end{aligned}$$

(c) Find the linear approximation of the function when $(x, y) \approx (2, 1)$.

$$f(x, y) \approx 9 + 5(x - 2) + 8(y - 1)$$

Problem 6. Consider a rectangular cardboard box with dimensions ℓ, w, h. If the box has no lid, then the surface area (i.e., the amount of cardboard) is given by $A = \ell w + 2\ell h + 2wh$.

(a) Compute the partial derivatives $\partial A/\partial \ell$, $\partial A/\partial w$ and $\partial A/\partial h$.

$$\begin{aligned}
\partial A/\partial \ell &= w + 2h, \\
\partial A/\partial w &= \ell + 2h, \\
\partial A/\partial h &= 2\ell + 2w.
\end{aligned}$$

(b) Suppose that the dimensions of the box are measured to be $\ell = 10 \pm 0.1$, $w = 5 \pm 0.1$ and $h = 3 \pm 0.1$, in centimeters. Use differentials to estimate the uncertainty in the surface area of the box.

$$\begin{aligned}
dA &= (\partial A/\partial \ell)d\ell + (\partial A/\partial w)dw + (\partial A/\partial h)dh \\
&= (w + 2h)d\ell + (\ell + 2h)dw + (2\ell + 2w)dh \\
&= (5 + 2 \cdot 3)(0.1) + (10 + 2 \cdot 3)(0.1) + (2 \cdot 10 + 2 \cdot 5)(0.1) \\
&= (11)(0.1) + (16)(0.1) + (30)(0.1) \\
&= (57)(0.1) \\
&= 5.7 \text{ cm}^2
\end{aligned}$$

$$A = 140 \pm 5.7 \text{ cm}^2$$