HW3 due <u>now</u>.

Today: Discuss HW3
Discuss FTLA.

Problems 1 & 2.

$$\begin{cases} x + 2y + 3z = 4, \\ 2x + 3y + 4z = 5, \\ 3x + 4y + 5z = 6. \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \end{array}\right) \begin{array}{l} R_1' = R_1 \\ R_2' = R_2 - 2R_1 \\ R_3' = R_3 - 3R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -2 & -4 & -6 \end{array}\right) \begin{array}{l} R_1'' = R_1' \\ R_2'' = -R_2' \\ R_3'' = R_3' \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} R_1''' = R_1'' \\ R_2''' = R_2'' \\ R_3''' = R_3'' - (-2)R_2'' \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} R_1'''' = R_1''' - 2R_2''' \\ R_2'''' = R_2''' \\ R_3'''' = R_3''' \end{array}$$

DONE.

$$\begin{cases} x + 0 - z = -2 \\ 0 + y + 2z = 3 \\ \cancel{0 = 0} \quad \text{redundant.} \end{cases}$$

Solution is a line.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Let's think about this in terms of matrices:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.$$

Nullspace:

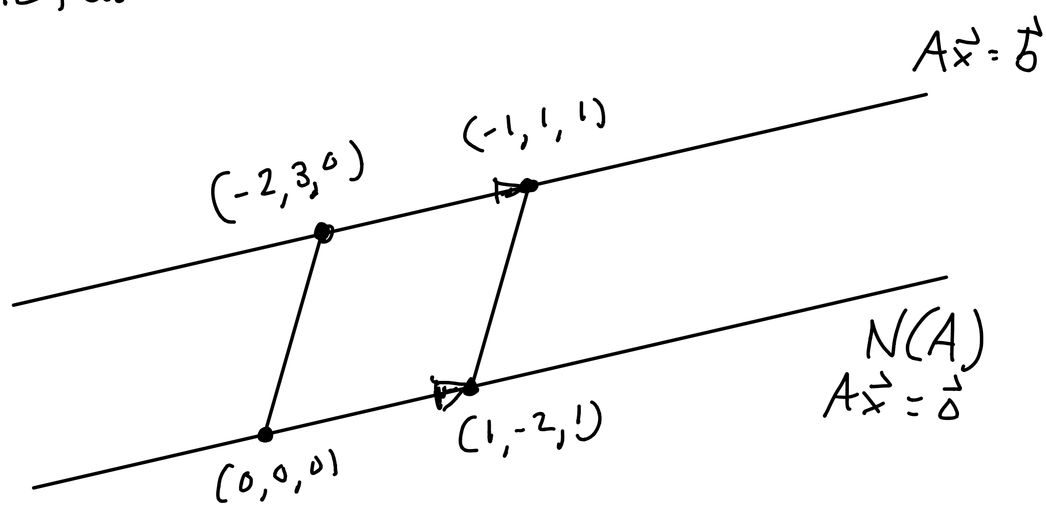
$N(A) = \{ \vec{x} : A\vec{x} = \vec{0} \}$.

Solve $A\vec{x} = \vec{0}$.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 0 \\ 3 & 4 & 5 & 0 \end{array}\right) \xrightarrow{\text{RREF}} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

it's a line: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

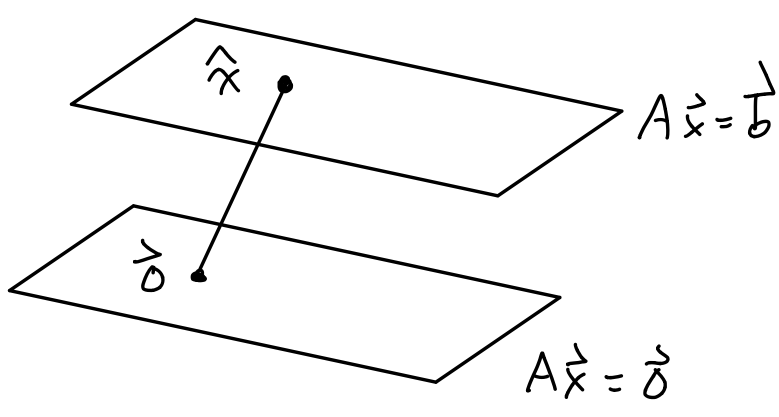
Picture:



We observe that the solutions to $A\vec{x} = \vec{b}$ are parallel to the nullspace.

<u>Theorem</u>: Let A be $m \times n$ matrix,
let $\vec{x}$ be $n \times 1$ vector of unknowns,
let $\vec{b}$ be $m \times 1$ vector of constants,
let $r = \text{rank}(A)$.
$= \#$ pivots in $\text{RREF}(A)$.

Then the solutions to $A\vec{x} = \vec{b}$ form an $(n-r)$-dimensional plane that is parallel to the nullspace:



If $\hat{x}$ is any particular solution, then the general solution is

$\hat{x} + N(A)$

$= \hat{x} +$ all solutions of $A\vec{x} = \vec{0}$.

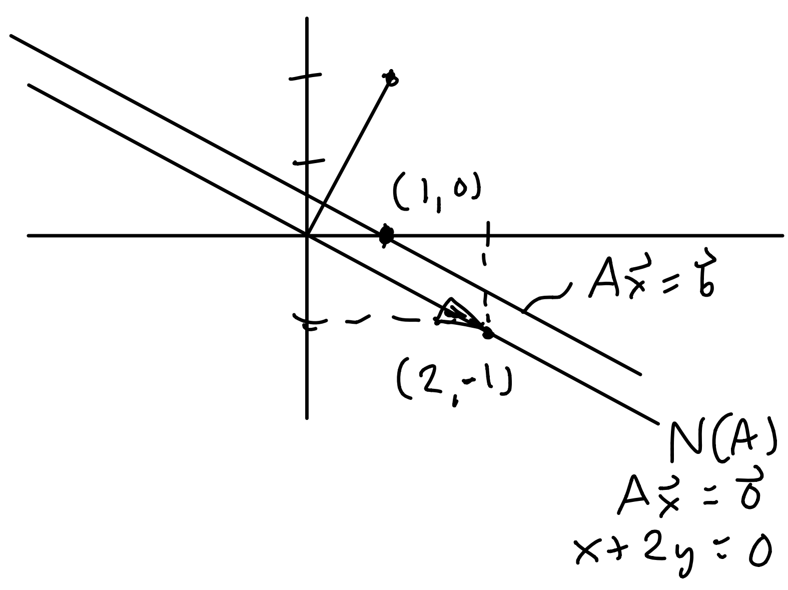
Jargon: A system of the form $A\vec{x} = \vec{0}$ is called "homogeneous."

The solution to a non-homogeneous system $A\vec{x} = \vec{b}$ is

$$\begin{pmatrix}\text{one particular} \\ \text{solution}\end{pmatrix} + \begin{pmatrix}\text{general homogeneous} \\ \text{solution}\end{pmatrix}$$

We've already seen this:

Let $A = (1, 2)$, $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\vec{b} = (1)$



$A\vec{x} = \vec{b}$ $\qquad\qquad$ $A\vec{x} = \vec{0}$

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}$$

$$x + 2y = 1 \qquad\qquad x + 2y = 0.$$

Note $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a particular solution of $A\vec{x} = \vec{b}$, and $N(A) = t\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

So general solution is the line

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + N(A) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Review of FTLA.

Let A be $m \times n$ matrix. We have a column space and row space:

$$C(A) \subseteq \mathbb{R}^m \qquad C(A) = R(A^T)$$

$$R(A) \subseteq \mathbb{R}^n \qquad R(A) = C(A^T).$$

FTLA:

$$\dim C(A) = \dim R(A)$$

$\#$ pivots in RREF(A) = $\#$ pivots in RREF(A^T).

This number is called the <u>rank of A</u>:

$$r = \text{rank}(A) = \# \text{ pivots}$$

Observe: $\quad 0 \leq r \leq \min\{m, n\}$

[The only matrices of rank zero are the "zero matrices."]

The solutions to $A\vec{x} = \vec{0}$ are called the null space:

$$N(A) = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\} \subseteq \mathbb{R}^n$$

$$\begin{aligned} \dim N(A) &= \# \text{ free variables} \\ &= \# \text{ nonpivot columns in } A \\ &= \# \text{ columns} - \# \text{pivots} \\ &= n - r. \end{aligned}$$

Furthermore, we have

$$N(A) = R(A)^{\perp}.$$

Indeed,

$$A\vec{x} = \vec{0} \iff \begin{pmatrix} \vec{r}_1 \cdot \vec{x} \\ \vec{r}_2 \cdot \vec{x} \\ \vdots \\ \vec{r}_m \cdot \vec{x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\iff \vec{r}_i \cdot \vec{x} = 0$ for every row $\vec{r}_i$ of matrix A.

$\iff \vec{x} \perp$ to every row of A.

In other words:

$$\vec{x} \in N(A) \iff \vec{x} \in R(A)^{\perp}.$$

$$N(A) = R(A)^{\perp}$$
$$N(A)^{\perp} = R(A).$$

Since $N(A), R(A) \subseteq \mathbb{R}^n$ this implies

$$\begin{aligned} \dim R(A) &= n - \dim N(A) \\ &= n - (n-r) \\ &= r. \end{aligned}$$

This is how we <u>prove</u> the FTLA.

To summarize:

(1) $\dim C(A) + \dim N(A) = n$

(2) $\dim R(A) + \dim N(A) = n$

Hence,

(3) $\dim C(A) = \dim R(A)$ ✓

(1) is sometimes called the

"Rank-Nullity Theorem"

$\dim C(A) =$ "rank"

$\dim N(A) =$ "nullity"

Finally, there is one more

"Fundamental subspace"

$$C(A)^{\perp} \subseteq \mathbb{R}^m$$

$$\begin{aligned} \dim C(A)^{\perp} &= ? \\ &= m - \dim C(A) \\ &= m - r. \end{aligned}$$

This space has another name:

$$C(A) = R(A^T)$$

$$C(A)^\perp = R(A^T)^\perp = N(A^T).$$

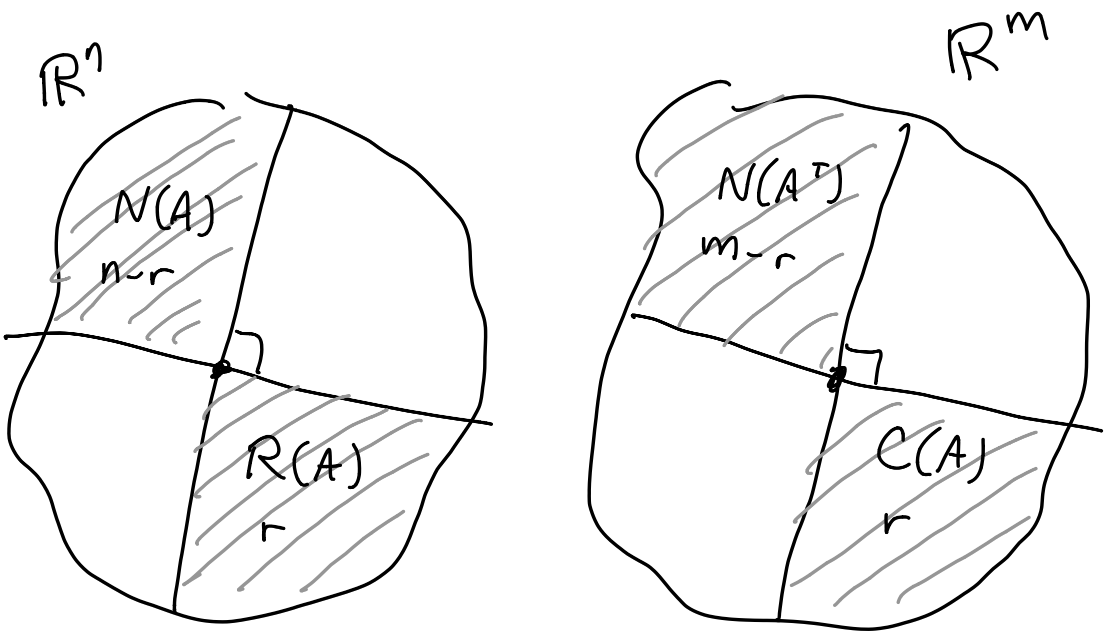
Emphasis: For any $\vec{y} \in \mathbb{R}^m$,

$$A^T \vec{y} = \vec{0} \iff \vec{y} \perp \text{ to every row of } A^T$$

$$\iff \vec{y} \perp \text{ to every } \underline{\text{column}} \text{ of } A$$

Gilbert Strang calls $N(A^T)$ the "left nullspace" of A.

The Big Picture:



Example:

$$A = \begin{pmatrix} \boxed{\begin{matrix} 1 & 1 \\ 2 & 3 \\ 1 & 0 \end{matrix}} & \begin{matrix} 2 \\ 1 \\ 5 \end{matrix} \end{pmatrix}, \quad A^T = \begin{pmatrix} \boxed{\begin{matrix} 1 & 2 \\ 1 & 3 \\ 2 & 1 \end{matrix}} & \begin{matrix} 1 \\ 0 \\ 5 \end{matrix} \end{pmatrix}.$$

$$RREF(A) = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}, \quad RREF(A^T) = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$r = \text{rank}(A) = \text{rank}(A^T) = 2$$

$$C(A) = \langle \text{pivot columns of } A \rangle.$$

$$= s\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \quad \text{plane}$$

$$R(A) = \langle \text{pivot columns of } A^T \rangle$$

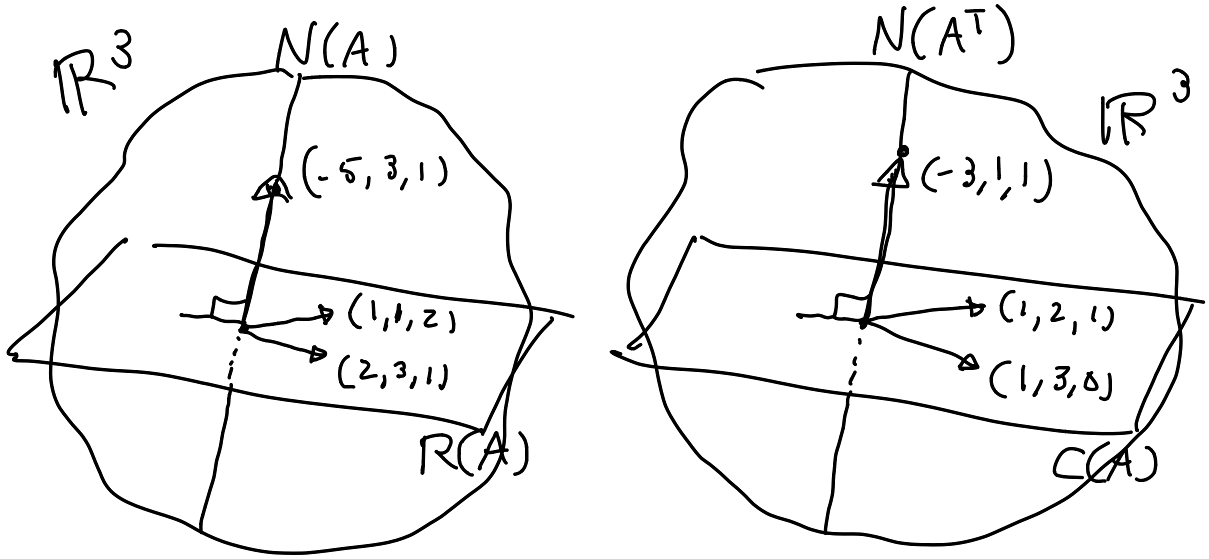
$$C(A^T) = s\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}. \quad \text{plane}$$

$$N(A) = R(A)^\perp$$

$$= t\begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix} \quad \text{line.}$$

$$N(A^T) = C(A)^{\perp}$$
$$= t\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

Picture:



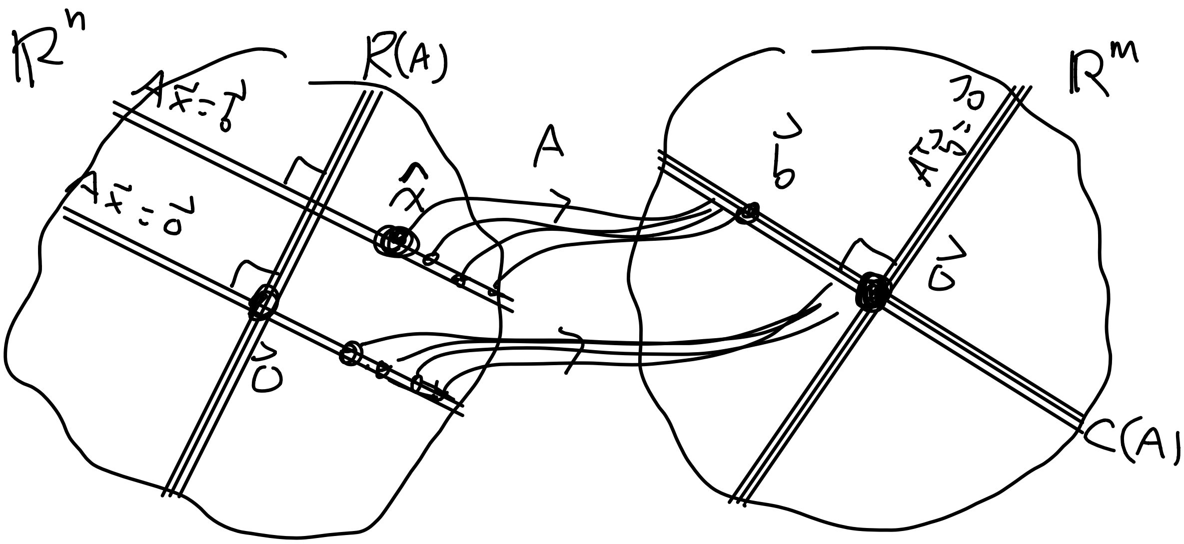
How does $A\vec{x} = \vec{b}$ fit into this picture?

- If $\vec{b}$ is not in $C(A)$ then there is NO SOLUTION!

Reason: The column space can be expressed as $\{A\vec{x} : \vec{x} \in \mathbb{R}^n\}$.

$$A\vec{x} = \underbrace{x_1\vec{c}_1 + x_2\vec{c}_2 + \cdots + x_n\vec{c}_n}_{\text{all linear combinations of the columns.}}$$

- If $\vec{b} \in C(A)$ then there is a solution. Say $A\hat{x} = \vec{b}$ is one particular solution. Then full solution is $\hat{x} + N(A)$.



Fat lines indicate high dimensional subspaces looked at "From the side."

NEVER MIND!

For Monday's Quiz 3:

- Compute RREF
- Interpret the RREF.
  - perhaps use it to solve a linear system
  - perhaps use it to find a basis for the column space. (i.e. the pivot columns)
- Memorize the basic data of FTLA:

$C(A) = R(A^T)$

$R(A) = C(A^T)$

$N(A) = R(A)^{\perp}$

And the dimensions:

$\dim C(A) = \dim R(A) = \mathbf{r}$

$= \#\text{pivots in RREF } (A \text{ or } A^T)$

$\dim N(A) = n - r$

$\dim N(A^T) = m - r.$