HW3 due tomorrow before class.

Recall from last time:

Given $m \times n$ matrix A

and $n \times 1$ column vector $\vec{x}$,

we define an $m \times 1$ column vector "$A\vec{x}$"

(say "A times $\vec{x}$") as follows.

Two Equivalent Descriptions.

- If $A = \underbrace{\begin{pmatrix} | & | & & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & & | \end{pmatrix}}_{}\Big\}m$

has column vectors $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n \in \mathbb{R}^m$,

then

$$A\vec{x} = A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{c}_1 + x_2\vec{c}_2 + \cdots + x_n\vec{c}_n$$

"a linear combination of the columns of A".

- If $A = \begin{pmatrix} \text{---} \; \vec{r}_1^{\,T} \; \text{---} \\ \vdots \\ \text{---} \; \vec{r}_m^{\,T} \; \text{---} \end{pmatrix}$

then we have

$$A\vec{x} = \begin{pmatrix} \vec{r}_1^{\,T}\vec{x} \\ \vec{r}_2^{\,T}\vec{x} \\ \vdots \\ \vec{r}_m^{\,T}\vec{x} \end{pmatrix} = \begin{pmatrix} \vec{r}_1 \cdot \vec{x} \\ \vec{r}_2 \cdot \vec{x} \\ \vdots \\ \vec{r}_m \cdot \vec{x} \end{pmatrix}.$$

To see that these are really the same, consider

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \end{pmatrix} \quad \& \quad \vec{x} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}.$$

By columns:

$$A\vec{x} = 2\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 4\begin{pmatrix} 2 \\ 0 \end{pmatrix} + (-1)\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -2 \end{pmatrix} + \begin{pmatrix} 8 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ -4 \end{pmatrix}.$$

By rows:

$$A\vec{x} = \begin{pmatrix} (1,2,3)\cdot(2,4,-1) \\ (-1,0,2)\cdot(2,4,-1) \end{pmatrix}$$

$$= \begin{pmatrix} 2+8-3 \\ -2+0-2 \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$$

SAME ✓

Today: The FTLA

"Fundamental Theorem of Linear Algebra" (See the cover of Gilbert Strang's Textbook.)

Recall: A subspace $U \subseteq \mathbb{R}^n$ is a d-plane passing through the origin.

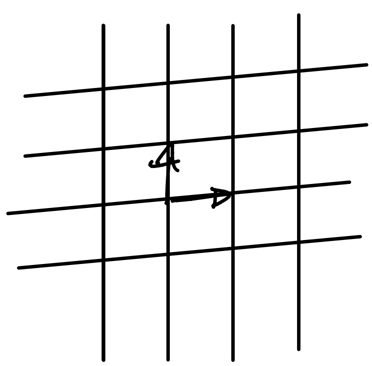
This means we can find some <u>basis</u>

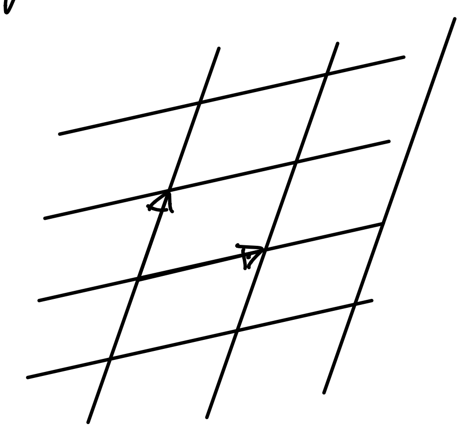
of independent vectors $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_d \in \mathbb{R}^n$ such that

$$U = \{ t_1 \vec{u}_1 + t_2 \vec{u}_2 + \dots + t_d \vec{u}_d : t_1, \dots, t_d \in \mathbb{R} \}$$

Note that the basis is not <u>unique</u>.

Example: $U = \mathbb{R}^2$. Then any two nonparallel vectors give a basis





Now consider some $m \times n$ matrix:

$$A = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix}}_{n} \Big\} m$$

Let's name the column vectors & row vectors:

$$A = \begin{pmatrix} | & | & & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_r \\ | & | & & | \end{pmatrix} = \begin{pmatrix} \text{---} \ \vec{r}_1^{\,T} \ \text{---} \\ \text{---} \ \vec{r}_2^{\,T} \ \text{---} \\ \\ \text{---} \ \vec{r}_m^{\,T} \ \text{---} \end{pmatrix}$$

where $\vec{c}_1, \dots, \vec{c}_n \in \mathbb{R}^m$

$\vec{r}_1, \dots, \vec{r}_m \in \mathbb{R}^n$.

There are "Four Fundamental Subspaces" associated to the matrix A:

- The <u>column space</u> $C(A) \subseteq \mathbb{R}^m$:

$$C(A) = \{ t_1 \vec{c}_1 + \cdots + t_n \vec{c}_n \} \subseteq \mathbb{R}^m$$

= all linear combinations of the columns of A.

- The <u>row space</u> $R(A) \subseteq \mathbb{R}^n$:

$$R(A) = \{ t_1 \vec{r}_1 + \cdots + t_m \vec{r}_m \} \subseteq \mathbb{R}^n$$

= all linear combinations of the rows of A.

[Remark: We have

$$R(A) = C(A^T) \subseteq \mathbb{R}^n$$

$$C(A) = R(A^T) \subseteq \mathbb{R}^m$$

where A^T is the <u>transpose</u> matrix.]

Example: $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{pmatrix}.$

$$C(A) = \left\{ t_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + t_3 \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} + t_4 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}$$

$$\subseteq \mathbb{R}^3$$

$$R(A) = \left\{ t_1 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} + t_3 \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} \right\} \subseteq \mathbb{R}^4$$

But Observe: $C(A) \subseteq \mathbb{R}^3$

means that $\dim C(A) \leq 3$.

At least one of the columns of A is <u>redundant</u>.

Goal: Find a basis for $C(A)$, i.e., find the redundant columns an throw them away.

Answer: Compute RREF.

$$\text{RREF}(A) = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

It's much easier to find redundant columns in the RREF:

$$\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = -1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

We see that 3rd & 4th columns are redundant. I claim that the <u>same</u> must be true for A. Indeed, we check that the same column relations hold in A:

$$\vec{c}_3 = -\vec{c}_1 + 2\vec{c}_2 \quad \& \quad \vec{c}_4 = -2\vec{c}_1 + 3\vec{c}_2$$

$$\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = -\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \checkmark \qquad \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 3\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \checkmark$$

In this case we conclude that the "pivot columns" $\vec{c}_1$ & $\vec{c}_2$ are a basis for the column space:

$$C(A) = \left\{ t_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right\} \subseteq \mathbb{R}^3$$

$$\dim C(A) = 2$$

In general, I claim that the pivot columns of $RREF(A)$ tell us which columns of A give a <u>basis</u> of the column space:

$$\dim C(A) = \#\text{ pivots in } RREF(A)$$

[Proof: Row operations preserve column relations.]

Now what about the row space? Well, since $R(A) = C(A^T)$ we can use the <u>same trick</u>!

Compute RREF(A^T):

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we find that

$$\vec{r}_3 = -\vec{r}_1 + 2\vec{r}_2$$

$$\begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} = -\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \checkmark$$

We conclude that $\vec{r}_3$ is redundant & that $\vec{r}_1, \vec{r}_2$ is a basis for the row space:

$$R(A) = \left\{ t_1 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \right\} \subseteq \mathbb{R}^4$$

"2D subspace of 4D space."

In conclusion, we have found that

$$\dim C(A) = \dim R(A) \qquad (=2)$$

I claim that the same equality holds for any matrix whatsoever.

<u>FTLA</u>: For any matrix A we have

$$\dim C(A) = \dim R(A).$$

This common dimension is called the <u>rank</u> of A. ///

Before explaining <u>why</u> this is true, let's see more examples.

- Random 3×3 matrix of row rank 1:

$$\begin{pmatrix} 1 & 6 & 9 \\ 2 & 12 & 18 \\ -1 & -6 & -9 \end{pmatrix} \quad \begin{array}{l} \vec{r}_1^{\,T} \\ \vec{r}_2^{\,T} = 2\vec{r}_1^{\,T} \\ \vec{r}_3^{\,T} = -\vec{r}_1^{\,T} \end{array}$$

The row space is just a <u>line</u>:

$$R(A) = t\begin{pmatrix}1\\6\\9\end{pmatrix}.$$

Now FTLA says

$$\dim C(A) = \dim R(A) = 1,$$

hence the column space should also be a line. Is it?

Indeed, we observe that

$$C(A) = t\begin{pmatrix}1\\2\\-1\end{pmatrix} \quad \checkmark$$

- Random 3×3 matrix of row rank 2:

$$\begin{array}{cc} \begin{pmatrix}1 & 4 & 2\\ 7 & 0 & 2\\ 8 & 4 & 4\end{pmatrix} & \begin{array}{l} \vec{r}_1^{\,T} \\ \vec{r}_2^{\,T} \\ \vec{r}_3^{\,T} = \vec{r}_1^{\,T} + \vec{r}_2^{\,T} \end{array} \end{array}$$

By construction $\vec{r}_1^{\,T}, \vec{r}_2^{\,T}$ is a basis for the row space:

$$R(A) = \left\{ t_1\begin{pmatrix}1\\4\\2\end{pmatrix} + t_2\begin{pmatrix}7\\0\\2\end{pmatrix} \right\} \subseteq \mathbb{R}^3.$$

Now FTLA says

$$\dim C(A) = \dim R(A) = 2,$$

hence there must be <u>some</u> relation among the columns. Is there?

Check:

$$\begin{pmatrix} 1 & 4 & 2 \\ 7 & 0 & 2 \\ 8 & 4 & 4 \end{pmatrix} \xrightarrow[\text{RREF}]{} \begin{pmatrix} 1 & 0 & 2/7 \\ 0 & 1 & 3/7 \\ 0 & 0 & 0 \end{pmatrix}$$

we observe that

$$\vec{c}_3 = \frac{2}{7}\vec{c}_1 + \frac{3}{7}\vec{c}_2$$

$$\begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = \frac{2}{7}\begin{pmatrix} 1 \\ 7 \\ 8 \end{pmatrix} + \frac{3}{7}\begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}$$

We say that $\vec{c}_3$ column is redundant and $\vec{c}_1, \vec{c}_2$ is a basis for $C(A)$:

$$C(A) = \left\{ t_1 \begin{pmatrix} 1 \\ 7 \\ 8 \end{pmatrix} + t_2 \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix} \right\}.$$

AMAZING !

Why does this happen?

The proof of FTLA involves a third fundamental subspace associated to a matrix.

- The nullspace of matrix A is the set of solutions to the equation $A\vec{x} = \vec{0}$:

$$N(A) = \{ \vec{x} : A\vec{x} = \vec{0} \}$$

If A is $m \times n$, then $\vec{x} \in \mathbb{R}^n$, hence $N(A) \subseteq \mathbb{R}^n$.

Now we will prove two basic facts about the nullspace, and the FTLA will follow immediately.

Facts:

1) $\dim N(A) + \dim C(A) = n$
(# columns)

2) $\dim N(A) + \dim R(A) = n$.

Then combining these gives

$$\cancel{\dim N(A)} + \dim C(A) = \cancel{\dim N(A)} + \dim R(A)$$

$$\dim C(A) = \dim R(A) \quad \checkmark$$

Proof of 1

Basis of $C(A)$ comes from the pivot columns of RREF(A), hence

$$\dim C(A) = \#\text{pivot columns}.$$

We also know that

$$\dim N(A) = \#\text{free variables}$$
$$= \#\text{non-pivot columns}.$$

Hence:

$\dim C(A) + \dim N(A)$

$= \#$ pivot cols $+ \#$ nonpivot cols

$= \#$ columns

$= n.$ ✓

Proof of 2

Note that $N(A) \subseteq \mathbb{R}^n$

$R(A) \subseteq \mathbb{R}^n$

I claim that these subspaces are orthogonal complements:

$$N(A) = R(A)^\perp$$

Then it will follow from HW 3.5 that $\dim N(A) + \dim R(A) = n$.

Why do we have

$$N(A) = R(A)^\perp \quad ?$$

This is pretty much just by definition! Indeed:

$$A\vec{x} = \vec{0}$$

$$\iff \begin{pmatrix} \vec{r}_1^{\,T} \cdot \vec{x} \\ \vdots \\ \vec{r}_m^{\,T} \cdot \vec{x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\iff \vec{r}_i^{\,T} \cdot \vec{x} = 0 \text{ for all } i.$$

$\iff$ $\vec{x}$ is $\perp$ to every row of A.

Emphasis:

$A\vec{x} = \vec{0} \iff \vec{x}$ is $\perp$ to <u>every row</u> of A.

This completes the proof of FTLA.

Discussion next time.