Today: More discussion!

Tomorrow: No class, but office hours as usual: 11:30 – 12:30

Fridy Noon: Final Project Due.

More discussion about "diagonalization":

Suppose $n \times n$ matrix A has some eigenvectors,

$$A\vec{u}_1 = \lambda_1 \vec{u}_1, \ \ldots, \ A\vec{u}_d = \lambda_d \vec{u}_d.$$

We can express these d vector equations as a single matrix equation:

$$\begin{pmatrix} | & & | \\ A\vec{u}_1 & \cdots & A\vec{u}_d \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \lambda_1\vec{u}_1 & \cdots & \lambda_d\vec{u}_d \\ | & & | \end{pmatrix}$$

$$A\begin{pmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_d \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_d \\ | & & | \end{pmatrix}\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix}$$

$$AU = U\underline{\Lambda}$$

where U is $n \times d$ matrix of eigenvectors & $\underline{\Lambda}$ is $d \times d$ "diagonal" matrix of eigenvalues.

Special Case: <u>If</u> we can find n independent eigenvectors $\vec{u}_1, \ldots, \vec{u}_n$ (i.e., an eigen<u>basis</u>) then U is square and invertible:

$$AU = U\underline{\Lambda}$$

$$A = U\underline{\Lambda}U^{-1}$$

$$U^{-1}AU = \underline{\Lambda}.$$

In this case we say that matrix is "diagonalizable."

Remarks:

- These are the best kind of matrices.

- A "randomly chosen" matrix will be diagonalizable.
- But non-diagonalizable matrices do exist, for example:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Check: $\det(A - \lambda I) = 0$

$$\det\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)^2 - 0 = 0$$

$$(1-\lambda)^2 = 0$$

Only one eigenvalue: $\lambda = 1$.

1-eigenspace:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 0 + y = 0 \\ 0 + 0 = 0 \end{cases}$$

pivot: y
free: x

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

1-eigenspace is the x-axis.

There are no other eigenspaces!

We cannot find a basis of eigenvectors, so we conclude that A is <u>not</u> diagonalizable:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \neq U \Lambda U^{-1}$$

Why is diagonalization good?

If $\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ is diagonal

then $\Lambda^m = \begin{pmatrix} \lambda_1^m & & 0 \\ & \ddots & \\ 0 & & \lambda_n^m \end{pmatrix}$ easy ✓

$$e^{\Lambda t} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} \checkmark$$

Furthermore, if $A = U \Lambda U^{-1}$ then

$$A^m = (U\Lambda \cancel{U^{-1}})(\cancel{U}\Lambda \cancel{U^{-1}}) \cdots (\cancel{U}\Lambda U^{-1})$$
$$= U \Lambda \Lambda \cdots \Lambda U^{-1}$$
$$= U \Lambda^m U^{-1}.$$

$$e^{At} = U e^{\Lambda t} U^{-1} \quad \text{(same reasoning)}$$

Finally, if $L = \begin{pmatrix} l_1 & & \\ & \ddots & \\ & & l_n \end{pmatrix}$ and

$$U = \begin{pmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_n^T \end{pmatrix}$$

then

$$U L U^{-1} = l_1 \vec{u}_1 \vec{v}_1^T + \cdots + l_n \vec{u}_n \vec{v}_n^T$$
$$= \text{weighted sum of rank 1 matrices.}$$

Putting these together allows us to explicitly compute matrix powers & exponentials.

Example: From HW5

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

$$A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

This is diagonalizable:

$$A\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}^{-1}$$

$$A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{-3}\begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$= \frac{5}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{(-1)}{3}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 2 & -1 \end{pmatrix}$$

Now we can easily compute A^n & e^{At}:

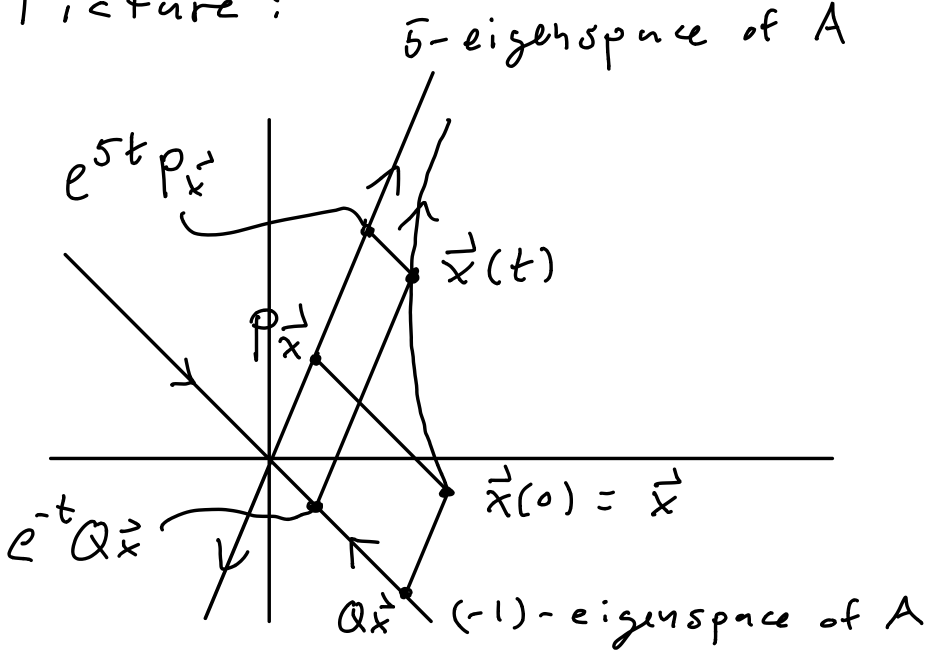
$$A^n = \frac{5^n}{3}\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} + \frac{(-1)^n}{3}\begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix}$$

$$e^{At} = \frac{e^{5t}}{3}\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} + \frac{e^{-t}}{3}\begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix}$$

$$= e^{5t}\, P + e^{-t}\, Q$$

projections

Picture:



Complex numbers:

$$\mathbb{C} = \{ a + bi : a, b \in \mathbb{R} \}$$

$i = \text{"}\sqrt{-1}\text{"}$, so that $i^2 = -1$.

These come up in linear algebra when we want to talk about rotations/oscillations.

Example: $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ = rotation by $90°$.

Is this diagonalizable?

$$\det(R - \lambda I) = 0$$

$$\det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$(-\lambda)^2 - (-1) = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda^2 = -1$$

$$\lambda^2 = \pm\sqrt{-1} = \pm i.$$

No real eigenvalues, but two "complex" eigenvalues. Is that ok? Sure, why not?

"i-eigenspace":

$$(R - iI)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Is this a "line"? Sure, why not? It's a line in "complex 2-dimensional space" $\mathbb{C}^2$.

"$(-i)$-eigenspace":

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

So <u>yes</u>, the matrix R <u>is</u> diagonalizable "over the complex numbers":

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = i\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Check!

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = -i\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{-1}$$

Then the exponential is

$$e^{\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \text{ ?}$$

This is related to Euler's formulas:

$$e^{it} = \cos t + i \sin t,$$

$$e^{-it} = \cos t - i \sin t,$$

$$e^{it} + e^{-it} = 2\cos t,$$

$$e^{it} - e^{-it} = 2i\sin t.$$

Whatever. The point is, if matrix A has complex eigenvalues then the dynamical system

$$\vec{x}'(t) = A\vec{x}(t)$$

will <u>oscillate</u>.

Prototype:

$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ has eigenvalues e^{it}, e^{-it}.

Diagonalization is sometimes called "spectral analysis." Why?

For complex vectors $\vec{x}, \vec{y} \in \mathbb{C}^n$ we define the dot product slightly differently:

$$\vec{x} \cdot \vec{y} = x_1^* y_1 + x_2^* y_2 + \cdots + x_n^* y_n$$

where $(a+ib)^* = (a-ib)$ is called "complex conjugation."

And the matrix transpose is replaced by the "conjugate transpose":

$$A = (a_{ij})$$

$$A^* = (a_{ji}^*) .$$

Beyond these two changes, the whole theory looks the same as before.

This is the correct setting to state an important theorem.

<u>The Spectral Theorem</u>:

("fundamental theorem of diagonalization").

Square matrix A has an orthonormal basis of eigenvectors if and only if

$$AA^* = A^*A .$$

In which case we can write

$$A = U \Lambda U^{-1}$$

where $U^{-1} = U^*$ <u>because</u> the columns $\vec{u}_j$ of U satisfy

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

Such a matrix $U^{-1} = U^*$ is called "unitary" (real version: "orthogonal").

This theorem tells us that many nice matrices are diagonalizable:

- $A^T = A$ symmetric.

Then $AA^T = AA = A^TA$ ✓

- $A^T = A^{-1}$ orthogonal.

Then $AA^T = AA^{-1}$

$$= I = A^{-1}A = A^TA$$ ✓

These are incredibly useful facts!

Why "spectral"?

Quantum physics (1920s):

- State of a quantum system is a "vector" in some ∞-dimensional "vector space" $\mathcal{H}$ over complex numbers.

- Observable quantity is a self-adjoint linear operator

$$A : \mathcal{H} \to \mathcal{H},$$

$$A^* = A.$$

- Possible values of the observation are the <u>eigenvalues</u> of A.

- Example: let A be energy/frequency of a photon emitted by a Hydrogen atom.

Eigenvalues of A =

the emission "spectrum" of Hydrogen.

A more modern application of linear algebra that has not become standard yet:

<u>Singular Value Decomposition</u> (SVD).

For any real matrix A $m \times n$ (say $n \leq m$) there exist orthogonal matrices

$$U^{-1} = U^T \quad m \times m$$
$$V^{-1} = V^T \quad n \times n$$

and non-negative real numbers

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0,$$

called the "singular values of A" such that

$$\underset{m \times n}{A} = \underset{m \times m}{U} \; \underset{m \times n}{\Sigma} \; \underset{n \times n}{V^T}$$

$$= \begin{pmatrix} \vec{u}_1 & \cdots & \vec{u}_m \end{pmatrix} \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \\ \hline & 0 & \end{pmatrix} \begin{pmatrix} \vec{v}_1^{\,T} \\ \vdots \\ \vec{v}_n^{\,T} \end{pmatrix}$$

$$= \underbrace{\sigma_1 \vec{u}_1 \vec{v}_1^T + \cdots + \sigma_n \vec{u}_n \vec{v}_n^T}_{\text{weighted sum of rank 1 matrices}} .$$

Eckart-Young Theorem:

Suppose $m \times n$ matrix A has full rank n (say $n \le m$). For any $r \le n$ let B be the rank r matrix that is closest to A:

$$\|A - B\|^2 \text{ minimized}$$

sum of squares of matrix entries.

Then $B = \underbrace{\sigma_1 \vec{u}_1 \vec{v}_1^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T}$

Sum includes only the r largest of the n singular values of A.

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Many applications:

- Total least squares, Principal component analysis in statistics.
- Image compression.

[Web Demonstration.]

That's All for Now!

To learn more, I recommend any textbook written by Gilbert Strang.