Tomorrow : Review

Thurs : No Class (Office Hours as usual, 11:30 — 12:30).

Friday : Final Project due noon 12pm.
(min 2 pages, max 10 pages)

Today : Final Discussion.

Recall the matrix $A = \begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix}$

satisfies

$$A\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \& \quad A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0.2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Suppose vectors $\vec{x}_0, \vec{x}_1, \vec{x}_2, \ldots$ are defined by

$$\vec{x}_0 = \begin{pmatrix} 100 \\ 0 \end{pmatrix} \quad \& \quad \vec{x}_{n+1} = A\vec{x}_n.$$

We saw that $\begin{pmatrix} 100 \\ 0 \end{pmatrix} = 25\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 75\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

and hence:

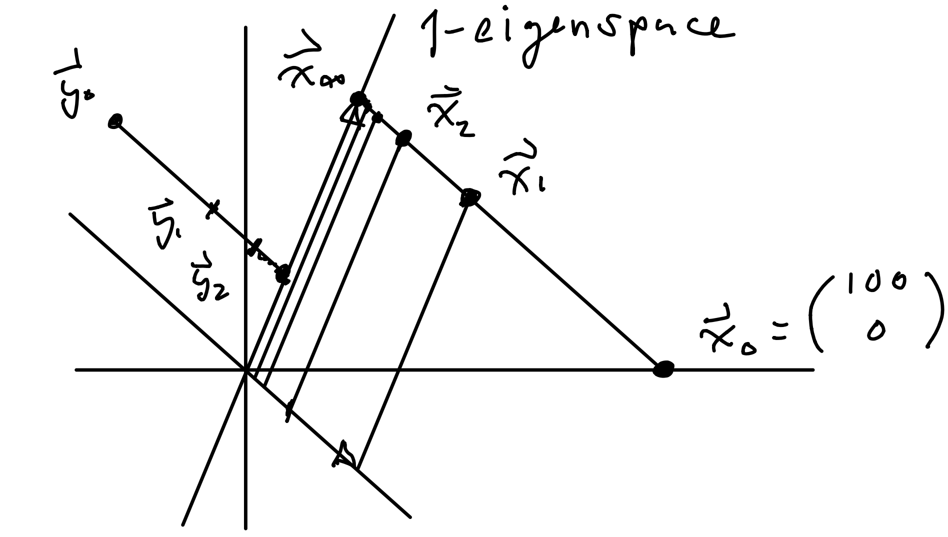
$$\vec{x}_n = A^n \vec{x}_0$$

$$= A^n \left[25 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 75 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right]$$

$$= 25 A^n \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 75 A^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= 25 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 75 (0.2)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

We can draw a <u>picture</u> of this:



$$\vec{x}_n = \begin{pmatrix} 25 \\ 75 \end{pmatrix} + (0.2)^n \begin{pmatrix} 75 \\ -75 \end{pmatrix} \longrightarrow \begin{pmatrix} 25 \\ 75 \end{pmatrix}$$

as $n \to \infty$. We could say

$$\text{“}\vec{x}_\infty\text{”} = \begin{pmatrix} 25 \\ 75 \end{pmatrix}$$

$$\text{“}\vec{x}_\infty\text{”} = \text{“}A^\infty\text{”} \begin{pmatrix} 100 \\ 0 \end{pmatrix}.$$

We could say that “A^∞” is the matrix that projects any point onto the line $t\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ in the direction of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

What is the matrix,

$$A^\infty = \,?$$

Example from HW5:

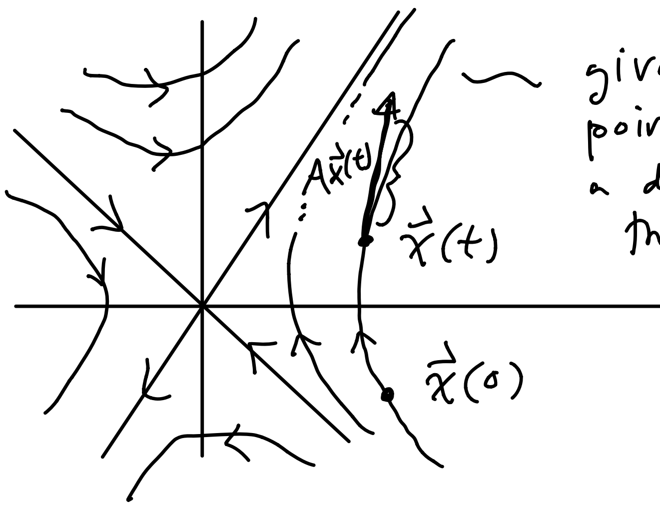
$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

$$A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Consider the differential equation

$$\vec{x}\,'(t) = A\,\vec{x}(t).$$

We can draw a picture:



given starting point $\vec{x}(0)$, find a description of this path.

I claim that for any starting point, the path is a "hyperbola." It's the eigenvectors and eigenvalues that tell us this:

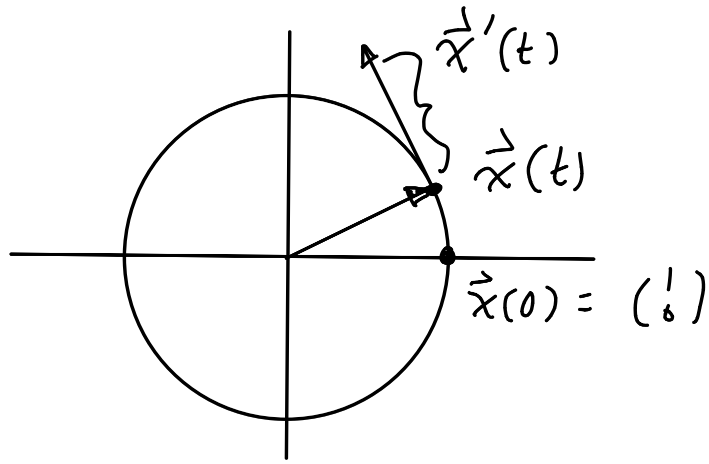
$$\vec{x}(t) = e^{At}\,\vec{x}(0)$$

We need the matrix: $e^{At} = ?$

One more:

Consider differential equation

$$\vec{x}'(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{x}(t)$$



Suppose we start at $\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Trajectory $\vec{x}(t) = ?$

GUESS: $\vec{x}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

CHECK: $\vec{x}'(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$

Does this work?

$$\vec{x}'(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{x}(t)$$

$$\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \checkmark$$

On the other hand, we know that

$$\vec{x}(t) = e^{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t} \vec{x}(0)$$

$$= e^{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

So $e^{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t} = ?$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 \frac{t^2}{2} + \cdots$$

[Taylor series for sin & cos:

$$\sin t = t - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots$$

$$\cos t = 1 - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots \quad]$$

$$= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Conclusion:

$$e^{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Why do we call it "diagonalization"?

$$AU = U\Lambda.$$

Let A be $m \times m$ matrix and suppose that we have some eigenvectors:

$$A\vec{u}_1 = \lambda_1 \vec{u}, \quad \ldots, \quad A\vec{u}_d = \lambda_d \vec{u}_d.$$

We can express these d vector equations as a single matrix equation:

$$A\begin{pmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_d \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_d \\ | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix}$$

jth column = jth column

$$A\vec{u}_j = \lambda_j \vec{u}_j \quad ✓$$

The Eigenvector Equation

$$\underset{m\times m}{A}\ \underset{m\times d}{U} = \underset{m\times d}{U}\ \underset{d\times d}{\Lambda}$$

where $U = \underbrace{\begin{pmatrix} \vec{u}_1 & \cdots & \vec{u}_d \end{pmatrix}}_{d} \Big\} m$

is an $m\times d$ matrix of eigenvectors,

and $\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix}$

is a $d\times d$ diagonal matrix of eigenvalues.

Suppose we can find m independent eigenvectors (an "eigenbasis").

Then U is $m\times m$ and invertible, so

$$AU = U\Lambda$$
$$A = U\Lambda U^{-1}$$

or

$$U^{-1} A U = \Lambda .$$

Then we say that the matrix A has been "diagonalized."

So what?

Theorem: Let $A = U \Lambda U^{-1}$. Then

- $A^n = U \Lambda^n U^{-1}$
- $f(A) = U f(\Lambda) U^{-1}$

  for any polynomial expression

  $f(x) = a_0 + a_1 x + a_2 x^2 + \dots$
- $e^{At} = U e^{\Lambda t} U^{-1}$.

And this is incredibly useful because

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}^n = \begin{pmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_m^n \end{pmatrix}$$:)

$$e^{\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix} t} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_m t} \end{pmatrix}.$$

This allows us to find explicit formulas for matrix powers and exponentials.

Examples :

$$\begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0.2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.2 \end{pmatrix}$$

$$\begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix}^n = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (0.2)^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (0.2)^n \end{pmatrix} \frac{1}{-4} \begin{pmatrix} -1 & -1 \\ -3 & 1 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (0.2)^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}.$$

$$= \frac{1}{4} \left[1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} + (0.2)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 3 & -1 \end{pmatrix} \right]$$

$$= \frac{1}{4} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{(0.2)^n}{4} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 3 & -1 \end{pmatrix}$$

sum of two "rank 1 matrices"

Note: As $n \to \infty$, $(0.2)^n \to 0$

so that

$$\begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix}^n \longrightarrow \frac{1}{4} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}.$$

This is the projection matrix that we were looking for :) .

Two loose ends:

- Proof of Theorem:

$$A = U \underline{\Lambda} U^{-1}$$

$$\begin{aligned} A^2 &= (U \underline{\Lambda} U^{-1})(U \underline{\Lambda} U^{-1}) \\ &= U \underline{\Lambda} (\cancel{U^{-1} U}) \underline{\Lambda} U^{-1} \\ &= U \underline{\Lambda} \underline{\Lambda} U^{-1} \\ &= U \underline{\Lambda}^2 U^{-1}. \end{aligned}$$

$$\begin{aligned} A^3 &= \cdots \\ &= U \underline{\Lambda}^3 U^{-1}. \end{aligned}$$

etc. ///

• Why is $\frac{1}{4}\begin{pmatrix}1\\3\end{pmatrix}(1\ \ 1)$ a projection?

For any vectors $\vec{a}, \vec{b} \in \mathbb{R}^n$, I claim that

$$P = \frac{1}{\underbrace{\vec{b}^T\vec{a}}_{\text{scalar}}} \underbrace{\vec{a}\,\vec{b}^T}_{n \times n \text{ matrix}}$$

is the projection onto the line $t\vec{a}$ in the direction(s) $\perp$ to $\vec{b}$.

Check:

$$P(t\vec{a}) = \frac{1}{\vec{b}^T\vec{a}}\,\vec{a}\,\vec{b}^T(t\vec{a})$$

$$= t\,\frac{1}{\cancel{\vec{b}^T\vec{a}}}\,\vec{a}\,(\cancel{\vec{b}^T\vec{a}}) = t\vec{a} \quad \checkmark$$

If $\vec{b}^T\vec{x} = 0$ then

$$P\vec{x} = \frac{1}{\vec{b}^T\vec{a}}\,\vec{a}\,(\underbrace{\cancel{\vec{b}^T\vec{x}}}_{0}) = \vec{0}. \quad \checkmark$$

[Discussion will continue tomorrow.]