HW5 due Monday

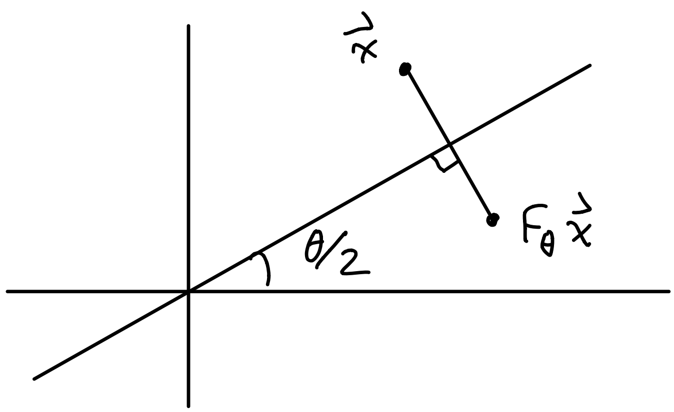
Quiz5 on Tuesday.

Current Topic: Diagonalization.

(eigenvalues & eigenvectors).

Conceptual Example:

$$F_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

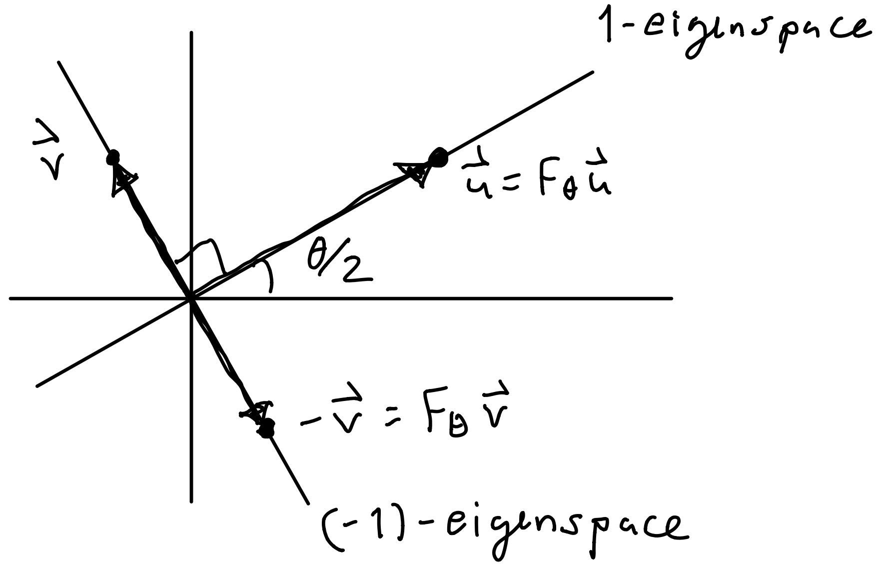


What are the eigenvalues & eigenvectors?

$$F_\theta \vec{u} = \lambda \vec{u} \quad \text{and} \quad \vec{u} \neq \vec{0}$$

Example: If $\vec{u}$ is <u>on</u> the line

then $F_\theta \vec{u} = \vec{u} = 1\vec{u}$:



Also, for any vector $\vec{v}$ $\perp$ to the line we have $F_\theta \vec{v} = -\vec{v} = (-1)\vec{v}$.

And this matrix has NO other eigenvectors.

Conclusion:

F_θ has eigenvalues 1 & -1.

1-eigenspace = line $t(\cos\frac{\theta}{2}, \sin\frac{\theta}{2})$

(-1)-eigenspace = line $t(\sin\frac{\theta}{2}, -\cos\frac{\theta}{2})$.

Remark: Sometimes we can learn everything about eigenvalues & eigenvectors by thinking about the geometry.

But sometimes we need to do calculations.

Example: Find all e.values & vectors of

$$A = \begin{pmatrix} 0 & 3 \\ 2 & 5 \end{pmatrix}.$$

[Recall: λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.]

$$\det(A - \lambda I) = 0$$

$$\det\left[\begin{pmatrix} 0 & 3 \\ 2 & 5 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = 0$$

$$\det\begin{pmatrix} -\lambda & 3 \\ 2 & 5-\lambda \end{pmatrix} = 0$$

$$(-\lambda)(5-\lambda) - 2\cdot 3 = 0$$

$$\lambda^2 - 5\lambda - 6 = 0$$

$$(\lambda - 6)(\lambda + 1) = 0 \quad \text{inspection}$$

or $\lambda = \dfrac{5 \pm \sqrt{25+24}}{2} = \dfrac{5 \pm 7}{2} = 6$ or -1

6 - Eigenspace:

$$(A - 6I)\vec{u} = \vec{0}$$

$$\begin{pmatrix} -6 & 3 \\ 2 & -1 \end{pmatrix}\vec{u} = \vec{0}$$

$$\left(\begin{array}{cc|c} -6 & 3 & 0 \\ 2 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} -6 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad x - \frac{1}{2}y = 0$$

$$\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2\, y \\ y \end{pmatrix} = y\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

(-1) - Eigenspace:

$$(A + 1I)\vec{v} = \vec{0}$$

$$\left(\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 6 & 0 \end{array}\right)$$

$$($$

$$\left(\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad x + 3y = 0$$

$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3y \\ y \end{pmatrix} = y \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

[Remark: Computations with 3×3 are worse, and in general we need a computer to find e.values & vectors of large square matrices]

What can we do with this?

Consider a linear recurrence equation:

$$\vec{x}_0 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad \& \quad \vec{x}_{n+1} = \begin{pmatrix} 0 & 3 \\ 2 & 5 \end{pmatrix} \vec{x}_n .$$

Find an explicit formula for the vector $\vec{x}_n$.

Step 1: Find e.values & e.vectors.

Pick your favorites:

$$\begin{pmatrix} 0 & 3 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 6\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 3 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = -1\begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad \checkmark$$

Step 2: Express the initial condition as a linear combination of eigenvectors:

$$\begin{pmatrix} 4 \\ 1 \end{pmatrix} = a\begin{pmatrix} 1 \\ 2 \end{pmatrix} + b\begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

By inspection, $a = 1$ & $b = 1$.

$$\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

Step 3: Solve. If $\vec{x}_{n+1} = A\vec{x}_n$

then $$\begin{aligned} \vec{x}_n &= A\vec{x}_{n-1} \\ &= AA\vec{x}_{n-2} \\ &\vdots \end{aligned}$$

$$\vdots$$

$$= AAA \cdots A\vec{x}_0$$

$$= A^n \vec{x}_0 .$$

In our case,

$$\vec{x}_n = \begin{pmatrix} 0 & 3 \\ 2 & 5 \end{pmatrix}^n \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 3 \\ 2 & 5 \end{pmatrix}^n \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 0 & 3 \\ 2 & 5 \end{pmatrix}^n \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ 2 & 5 \end{pmatrix}^n \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$= 6^n \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (-1)^n \begin{pmatrix} 3 \\ -1 \end{pmatrix} .$$

Done

Why did it work ?

For any $A\vec{u} = \lambda \vec{u}$ we have

$$A^n \vec{u} = A^{n-1} A \vec{u}$$

$$= A^{n-1} (\lambda \vec{u})$$

$$
\begin{aligned}
&= \lambda A^{n-1} \vec{u} \\
&= \lambda \lambda A^{n-2} \vec{u} \\
&= \lambda \lambda \cdots \lambda \vec{u} \\
&= \lambda^n \vec{u}. \qquad \checkmark
\end{aligned}
$$

What about

$$(A^3 - 2A^2 + A - 5I)\vec{u} \quad ?$$

$$
\begin{aligned}
&= A^3\vec{u} - 2A^2\vec{u} + A\vec{u} - 5\vec{u} \\
&= \lambda^3\vec{u} - 2\lambda^2\vec{u} + \lambda\vec{u} - 5\vec{u} \\
&= (\lambda^3 - 2\lambda + \lambda - 5)\vec{u}.
\end{aligned}
$$

More generally, for any polynomial expression

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d$$

we can define the <u>matrix</u>

$$f(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_d A^d.$$

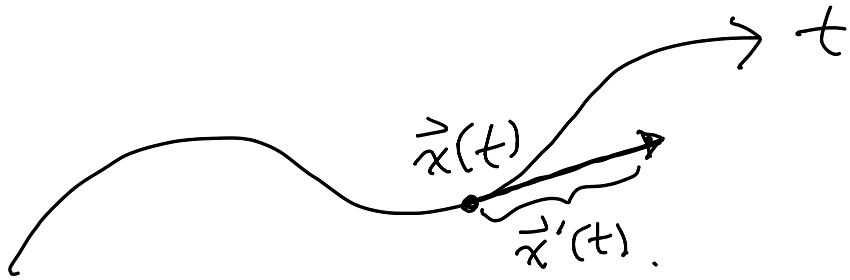
Then for any $A\vec{u} = \lambda \vec{u}$ we find

$$\underbrace{f(A)}_{\text{matrix}} \vec{u} = \underbrace{f(\lambda)}_{\text{scalar}} \vec{u} .$$

I will show you an amazing application of this.

Vector Fields.

Let $\vec{x}(t)$ be a parametrized path in $\mathbb{R}^n$:



The derivative is the instantaneous velocity at time t.

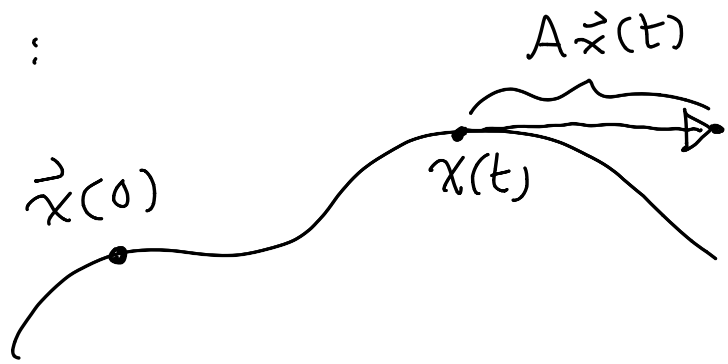
$$\vec{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$$

$$\vec{x}'(t) = (x_1'(t), x_2'(t), \ldots, x_n'(t)).$$

A vector field tells us a vector at every point. A "linear vector field" has the form:

$$\vec{x}'(t) = A\vec{x}(t).$$

Idea:



Our problem: Given a starting point $\vec{x}(0)$, find the unique path $\vec{x}(t)$ such that

$$\vec{x}'(t) = A\vec{x}(t)$$

Such a path is called a <u>flow line</u> of the vector field. The solution to this problem is <u>surpisingly nice</u>.

<u>Theorem</u>: For any square matrix A, the "linear system of ordinary differential equations with constant coefficients"

$$\vec{x}'(t) = A\vec{x}(t)$$

has the <u>unique</u> solution

$$\boxed{\vec{x}(t) = e^{At}\vec{x}(0)}$$

where e^{At} is the square matrix defined as follows:

$$e^{At} = I + At + A^2\frac{t^2}{2} + A^3\frac{t^3}{6} + \cdots$$

$$= \sum_{k=0}^{\infty} A^k \frac{t^k}{k!}$$

It turns out that this series always converges!

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Before we try to apply this, it's easy to see why it works:

$$\frac{d}{dt}\left(e^{At}\right) = \frac{d}{dt}\left(I + At + A^2\frac{t^2}{2} + A^3\frac{t^3}{6} + \cdots\right)$$

$$= 0 + A + A^2\frac{2t}{2} + A^3\frac{3t^2}{6} + \cdots$$

$$= A + A^2 t + A^3\frac{t^2}{2} + A^4\frac{t^3}{6} + \cdots$$

$$= A\left(I + At + A^2\frac{t^2}{2} + A^3\frac{t^3}{6} + \cdots\right)$$

$$= \underbrace{A}_{\text{matrix}}\,\underbrace{e^{At}}_{\text{matrix}}.$$

So we observe that $\vec{x}(t) = e^{At}\vec{x}(0)$ satisfies:

$$\vec{x}(0) = e^{0}\vec{x}(0) = I\vec{x}(0) \quad \checkmark$$

$$\vec{x}\,'(t) = \frac{d}{dt}\left(e^{At}\vec{x}(0)\right)$$

$$= Ae^{At}\vec{x}(0) = A\vec{x}(t) \quad \checkmark$$

You might worry that the matrix

$$e^{At} = I + At + A^2\frac{t^2}{2} + \cdots$$

is hard to compute. The good news is we don't need to compute it. We just need to know the eigenvalues & eigenvectors, and the following theorem makes this easy.

Theorem: If $A\vec{u} = \lambda\vec{u}$ for some square matrix A, then

$$\underbrace{e^{At}}_{\text{matrix}}\vec{u} = \underbrace{e^{\lambda t}}_{\text{scalar}}\vec{u}.$$

That is, the matrix e^{At} has <u>the same eigenvectors as A</u>. Only the eigenvalues change:

λ is an e.value of A $\iff$ $e^{\lambda t}$ is an e.value of e^{At}.

Let's apply this.
Suppose that functions $x(t)$, $y(t)$ satisfy the following initial conditions and differential equations:

$$\begin{cases} x(0) = 4 \\ y(0) = 1 \end{cases} \quad \begin{cases} x'(t) = 3y(t) \\ y'(t) = 2x(t) + 5y(t). \end{cases}$$

Find explicit formulas for $x(t)$ & $y(t)$.

I claim that we have already done all the work!

Define $\vec{x}(t) = (x(t), y(t))$.

so that $\vec{x}'(t) = A\vec{x}(t)$

has the solution

$$\vec{x}(t) = e^{At}\vec{x}(0).$$

In our case:

$$A = \begin{pmatrix} 0 & 3 \\ 2 & 5 \end{pmatrix}$$

has e.values & e.vectors

$$A\vec{u} = 6\vec{u} \quad \& \quad A\vec{v} = -1\vec{v}$$

$$A\begin{pmatrix}1\\2\end{pmatrix} = 6\begin{pmatrix}1\\2\end{pmatrix} \quad \& \quad A\begin{pmatrix}3\\-1\end{pmatrix} = -\begin{pmatrix}3\\-1\end{pmatrix}$$

Express the initial condition in terms of these:

$$\vec{x}(0) = \begin{pmatrix}4\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}3\\-1\end{pmatrix} = \vec{u} + \vec{v}.$$

Therefore, the solution is:

$$\begin{aligned}\vec{x}(t) &= e^{At}\vec{x}(0)\\ &= e^{At}(\vec{u}+\vec{v})\\ &= e^{At}\vec{u} + e^{At}\vec{v}\\ &= e^{6t}\vec{u} + e^{-t}\vec{v}\\ &= e^{6t}\begin{pmatrix}1\\2\end{pmatrix} + e^{-t}\begin{pmatrix}3\\-1\end{pmatrix}\end{aligned}$$

Conclusion:

$$x(t) = e^{6t} + 3e^{-t}$$

$$y(t) = 2e^{6t} - e^{-t}.$$

See how easy that was?
Well, at least it's no harder than solving linear recurrence equations.

$$\vec{x}_{n+1} = A\vec{x}_n \qquad \vec{x}'(t) = A\vec{x}(t)$$

Same method of solution.

Namely,
Step 1: Find all e.values & e.vectors of A:

$$A\vec{u}_1 = \lambda_1\vec{u}_1,\ A\vec{u}_2 = \lambda_2\vec{u}_2, \ldots, A\vec{u}_d = \lambda_d\vec{u}_d.$$

Step 2: Express initial conditions in terms of eigenvectors:

$$\vec{x}_0 = a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_d\vec{u}_d$$

$$\vec{x}(0) = b_1\vec{u}_1 + b_2\vec{u}_2 + \cdots + b_d\vec{u}_d.$$

Step 3: Solve.

$$\vec{x}_n = A^n\vec{x}_0$$

$\|$

$$= A^n\left(a_1\vec{u}_1 + \cdots + a_d\vec{u}_d\right)$$
$$= a_1 A^n\vec{u}_1 + \cdots + a_d A^n\vec{u}_d$$
$$= a_1\lambda_1^n\vec{u}_1 + \cdots + a_d\lambda_d^n\vec{u}_d$$

Done.

$$\vec{x}(t) = e^{At}\vec{x}(0)$$
$$= e^{At}\left(b_1\vec{u}_1 + \cdots + b_d\vec{u}_d\right)$$
$$= b_1 e^{At}\vec{u}_1 + \cdots + b_d e^{At}\vec{u}_d$$
$$= b_1 e^{\lambda_1 t}\vec{u}_1 + \cdots + b_d e^{\lambda_d t}\vec{u}_d.$$

Done.

This is the power of diagonalization. See HW5.5.