HW5 due before class on <u>Monday</u>.

Quiz 5 on T<u>uesdy</u>.

HW5 Problem 2 typo:

(a) $P = \vec{a}\,(\vec{a}^T\vec{a})^{-1}\,\vec{a}^T$

If $\vec{a}$ has shape 3×1, e.g., $\vec{a} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

then P has what shape?

$$\underset{1\times 3}{\vec{a}^T}\ \underset{3\times 1}{\vec{a}} = \underset{1\times 1}{\vec{a}\cdot\vec{a}} = \|\vec{a}\|^2.$$ Just a scalar!

$$P = \vec{a}\left(\frac{1}{\|\vec{a}\|^2}\right)\vec{a}^T$$

$$= \frac{1}{\|\vec{a}\|^2}\ \underset{3\times \cancel{1}\ \cancel{1}\times 3}{\vec{a}\,\vec{a}^T} = 3\times 3 \text{ matrix.}$$

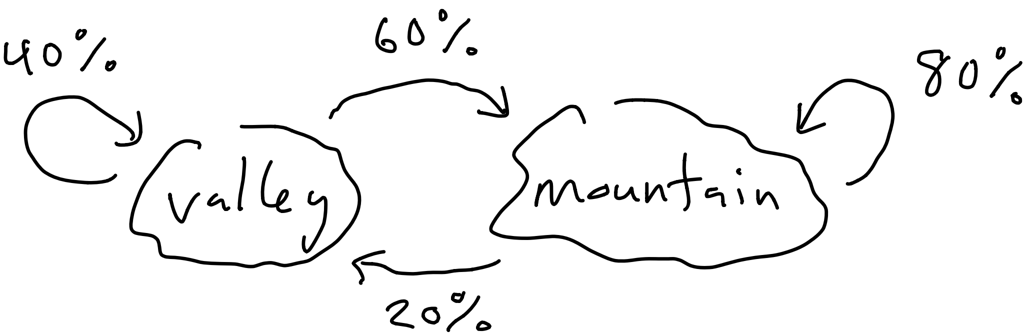
What I originally wrote:

$$P = \frac{1}{\|\vec{a}\|^2}\ \underset{3\times 1\ 3\times 1}{\vec{a}\,\vec{a}} = \text{NOT DEFINED!}$$

OOPS

Today: "Diagonalization" means finding the "correct" coordinate system for a given problem.

Example from last time: Yearly Migration of Bears



If x_n, y_n are # bears in the valley and on the mountain in year n, then

$$\begin{cases} x_{n+1} = 0.4\,x_n + 0.2\,y_n, \\ y_{n+1} = 0.6\,x_n + 0.8\,y_n. \end{cases}$$

We can express this system as a single matrix equation:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\vec{x}_{n+1} = A \vec{x}_n .$$

To analyze this problem we need to find the correct coordinate system, which is given by the "eigenvectors" of the matrix A.

[German: "eigen" = "belonging to."]

Definition: Let A be a SQUARE matrix. We say that a scalar $\lambda \in \mathbb{R}$ is an "eigenvalue" of A if there exists some nonzero vector $\vec{u} \neq \vec{0}$ such that

$$\boxed{A\vec{u} = \lambda \vec{u}}$$

In this case we say that $\vec{u}$ is an "eigenvector" of A, belonging to eigenvalue λ. We could also say $\vec{u}$ is a "λ-eigenvector" of A. ///

Before discussing how to compute eigenvalues/vectors, let me just <u>tell</u> you that the bear matrix A satisfies

$$\begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0.4 + 0.6 \\ 0.6 + 2.4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Say $\vec{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is a "1-eigenvector" of A:

$$A\vec{u} = 1\vec{u}.$$

AND:

$$\begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.4 + 0.2 \\ -0.6 + 0.8 \end{pmatrix} = \begin{pmatrix} -0.2 \\ 0.2 \end{pmatrix}$$

$$\begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0.2\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$A\vec{v} = (0.2)\,\vec{v}.$$

So $\vec{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is a "0.2-eigenvector" of A.

I claim that the equations

$$A\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix} \quad \& \quad A\begin{pmatrix}-1\\1\end{pmatrix} = 0.2\begin{pmatrix}-1\\1\end{pmatrix}$$

tell us everything we need to solve the problem.

Suppose we begin with

$x_0 = 100$ bears in the valley

$y_0 = 0$ bears on the mountain.

Then $\begin{pmatrix}x_n\\y_n\end{pmatrix} = A^n\begin{pmatrix}x_0\\y_0\end{pmatrix} = A^n\begin{pmatrix}100\\0\end{pmatrix}$

Now what? Express the initial state vector $\vec{x}_0 = \begin{pmatrix}100\\0\end{pmatrix}$ in terms of the "good coordinate system"

$$\vec{u} = \begin{pmatrix}1\\3\end{pmatrix} \quad \& \quad \vec{v} = \begin{pmatrix}-1\\1\end{pmatrix}.$$

In other words:

$$\vec{x}_0 = a\vec{u} + b\vec{v}$$

$$\begin{pmatrix}100\\0\end{pmatrix} = a\begin{pmatrix}1\\3\end{pmatrix} + b\begin{pmatrix}-1\\1\end{pmatrix}.$$

Find a & b:

$$\begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 100 \\ 3 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & -1 & 100 \\ 0 & 4 & -300 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & -1 & 100 \\ 0 & 1 & -75 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 0 & 25 \\ 0 & 1 & -75 \end{array}\right) \qquad \begin{array}{l} a = 25 \\ b = -75. \end{array}$$

Check:

$$\begin{pmatrix} 100 \\ 0 \end{pmatrix} = 25\begin{pmatrix} 1 \\ 3 \end{pmatrix} - 75\begin{pmatrix} -1 \\ 1 \end{pmatrix} \checkmark$$

Now that we have expressed the initial conditions in terms of eigenvectors we are done!

Solution:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \begin{pmatrix} 100 \\ 0 \end{pmatrix}$$

$$= A^n \left[25 \begin{pmatrix} 1 \\ 3 \end{pmatrix} - 75 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$$

$$= 25 A^n \begin{pmatrix} 1 \\ 3 \end{pmatrix} - 75 A^n \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

(*)

$$= 25 \begin{pmatrix} 1 \\ 3 \end{pmatrix} - 75 (0.2)^n \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

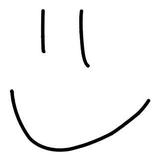
$$= \begin{pmatrix} 25 + 75(0.2)^n \\ 75 - 75(0.2)^n \end{pmatrix}.$$

x_n = # valley bears in year n

$= 25 + 75(0.2)^n$

y_n = # mountain bears in year n

$= 75 - 75(0.2)^n$.



Key Fact that made it work:

If $A\vec{u} = \lambda\vec{u}$ then $A^n\vec{u} = \lambda^n\vec{u}$ for any power n. $(*)$

How to ~~compute~~ eigenvalues/vectors?

Let A be square. Then λ is an eigenvalue

$\iff$ $A\vec{u} = \lambda\vec{u}$ for some $\vec{u} \neq \vec{0}$

$\iff$ $A\vec{u} - \lambda\vec{u} = \vec{0}$ for some $\vec{u} \neq \vec{0}$.

$\iff$ $\cancel{(A-\lambda)\vec{u} = \vec{0}}$

OOPS: $A-\lambda$ makes no sense!

$\iff$ $(\underbrace{A-\lambda I}_{\text{ok ✓}})\vec{u} = \vec{0}$ for some $\vec{u} \neq \vec{0}$.

HW4.4

$\iff$ $(A-\lambda I)^{-1}$ does not exist

Can we turn this statement into a useful equation?

Recall: We have several equivalent conditions for invertibility of a square $n \times n$ matrix A:

$$A^{-1} \text{ exists} \iff \text{rank}(A) = n$$
$$\iff \text{rows are independent}$$
$$\iff \text{columns are independent}$$
$$\iff N(A) = \{\vec{0}\}$$
$$\iff C(A) = \mathbb{R}^n$$
$$\iff \boxed{\det(A) \neq 0}$$

This last condition is useful for computing eigenvalues. Let's see:

λ is an eigenvalue of A

$$\iff (A - \lambda I)^{-1} \text{ does \underline{not} exist}$$

$$\iff \det(A - \lambda I) = 0.$$

Equations are good because they give us something to solve. ///

Let's use this to find the eigenvalues of the bear matrix.

$$A = \begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.4 - \lambda & 0.2 \\ 0.6 & 0.8 - \lambda \end{pmatrix}.$$

$$\det(A - \lambda I) = (0.4 - \lambda)(0.8 - \lambda) - (0.2)(0.6)$$

$$= 0.32 - 1.2\lambda + \lambda^2 - 0.12$$

$$= \lambda^2 - 1.2\lambda + 0.2$$

So the eigenvalues of A are the solutions of the "characteristic equation":

$$\det(A - \lambda I) = 0$$

$$\lambda^2 - 1.2\lambda + 0.2 = 0.$$

Wait a minute. This is a nonlinear equation! Correct. The computation of eigenvalues is a nonlinear problem.

Quadratic Formula:

$$\lambda = \frac{1.2 \pm \sqrt{(1.2)^2 - 4(0.2)}}{2}$$

$$= \frac{1.2 \pm \sqrt{1.44 - 0.8}}{2}$$

$$= \frac{1.2 \pm \sqrt{0.64}}{2}$$

$$= \frac{1.2 \pm 0.8}{2}$$

$$= \frac{1.2 + 0.8}{2} \text{ or } \frac{1.2 - 0.8}{2}$$

$$= 1 \text{ or } (0.2) \quad \checkmark$$

Finally, we can use the eigenvalues to find the eigenvectors.

Remark: If λ is an eigenvector of square $n \times n$ matrix A, then the λ-eigenvectors of A form a subspace of $\mathbb{R}^n$ called the "λ-eigenspace."

$$N(A - \lambda I) = \text{"}\lambda\text{-eigenspace" of } A$$
$$= \{ \vec{u} \in \mathbb{R}^n : (A - \lambda I)\vec{u} = \vec{0} \}$$
$$= \{ \vec{u} \in \mathbb{R}^n : A\vec{u} = \lambda \vec{u} \}. \quad ///$$

For the bears matrix, first compute the "1-eigenspace":

$$N(A - 1I)$$

$$N\begin{pmatrix} 0.4 - 1 & 0.2 \\ 0.6 & 0.8 - 1 \end{pmatrix}$$

$$\begin{pmatrix} 0.4 - 1 & 0.2 \\ 0.6 & 0.8 - 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 0.4-1 & 0.2 & 0 \\ 0.6 & 0.8-1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} -0.6 & 0.2 & 0 \\ 0.6 & -0.2 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} -0.6 & 0.2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Not a surprise because we know $A-1I$ is not invertible!

$$\left(\begin{array}{cc|c} 1 & -1/3 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$x - \tfrac{1}{3}y = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/3\, y \\ y \end{pmatrix} = y\begin{pmatrix} 1/3 \\ 1 \end{pmatrix}$$

Conclusion: The "1-eigenspace" is the line $t(\frac{1}{3}, 1)$. Check:

$$\begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} t\begin{pmatrix} 1/3 \\ 1 \end{pmatrix} = \underline{1} \cdot t\begin{pmatrix} 1/3 \\ 1 \end{pmatrix} \quad \checkmark$$

Note that $\vec{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is on this line.

[I chose $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ instead of $\begin{pmatrix} 1/3 \\ 1 \end{pmatrix}$ because I don't like fractions.]

Next compute the "0.2-eigenspace":

$$N(A - 0.2I)$$

$$\left(\begin{array}{cc|c} 0.4-0.2 & 0.2 & 0 \\ 0.6 & 0.8-0.2 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 0.2 & 0.2 & 0 \\ 0.6 & 0.6 & 0 \end{array}\right)$$

Surprise? NO. We knew that the rows were going to be parallel.

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$x + y = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

The "0.2-eigenspace" is the line $t(-1,1)$. Pick our favorite nonzero vector on this line:

$$\vec{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \checkmark$$

This completes the problem of the migrating bears.

Yes, it took the entire lecture, but now that we know the method we could solve a similar problem <u>much more quickly</u>.

NEXT TIME!