HW5 due Mon before class,

Quiz 5 beginning of Tues class.

Next Wed: Review

Next Thurs: No Class

Next Fri: Final Project due.

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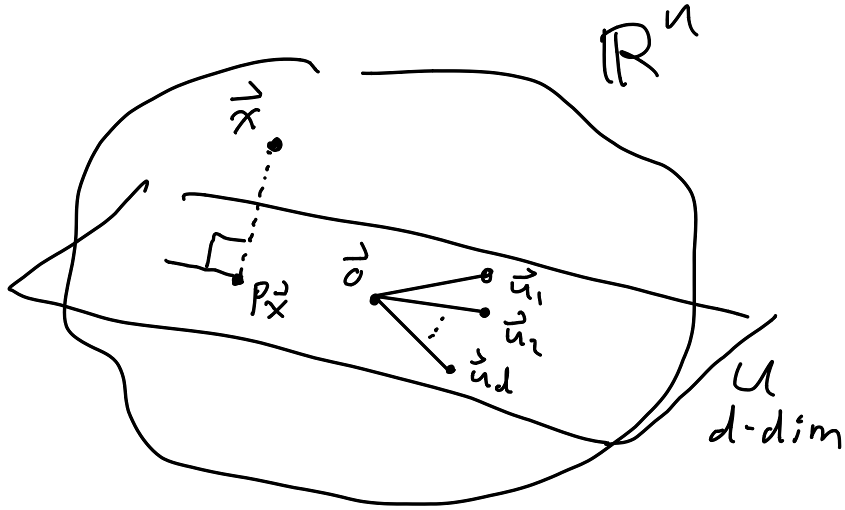
Review: Let $U \subseteq \mathbb{R}^n$ be a d-dim subspace of n-dim space:




Goal: For any point $\vec{x} \in \mathbb{R}^n$, find the closest point $P\vec{x}$ in the subspace.

Method: Define $n \times d$ matrix

$$A = n\left\{\underbrace{\begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_d \\ | & | & & | \end{pmatrix}}_{d}\right.$$

Note that

- $C(A) = U$.
- Columns $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_d$ are independent.

Then I claim that the projection matrix is given by

$$P_U = A(A^T A)^{-1} A^T.$$

Proof: Since $P\vec{x}$ is in $U$, we can write $\boxed{P\vec{x} = A\vec{y}}$ for some $\vec{y} \in \mathbb{R}^n$.

[Every point in $U = C(A)$ has the form $A$(some vector).]

Also we know that vector $P\vec{x} - \vec{x}$ is $\perp$ to $U$, i.e., is $\perp$ to every column of $A$. There is a quick way to say this :

$$\boxed{A^T(P\vec{x} - \vec{x}) = \vec{0}}$$

" $P\vec{x} - \vec{x}$ is $\perp$ to every row of $A^T$,
i.e., to every column of $A$ "

Combining the two boxed formulas gives

$$A^T(P\vec{x} - \vec{x}) = \vec{0}$$

$$A^T(A\vec{y} - \vec{x}) = \vec{0}$$

$$A^TA\vec{y} - A^T\vec{x} = \vec{0}$$

$$A^TA\vec{y} = A^T\vec{x}$$

$$\vec{y} = (A^TA)^{-1}A^T\vec{x}$$

$$A\vec{y} = A(A^TA)^{-1}A^T\vec{x}$$

$$P\vec{x} = A(A^TA)^{-1}A^T\vec{x}$$

Since this true for any $\vec{x}$, we have

$$P = A(A^TA)^{-1}A^T.$$

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Special Case: Project onto line $t\vec{a}$.

line $= C(\vec{a})$.

$$P = \vec{a}\,(\vec{a}^T\vec{a})^{-1}\,\vec{a}^T$$

scalar

$$= \frac{1}{\vec{a}\circ\vec{a}}\,\vec{a}\,\vec{a}^T$$

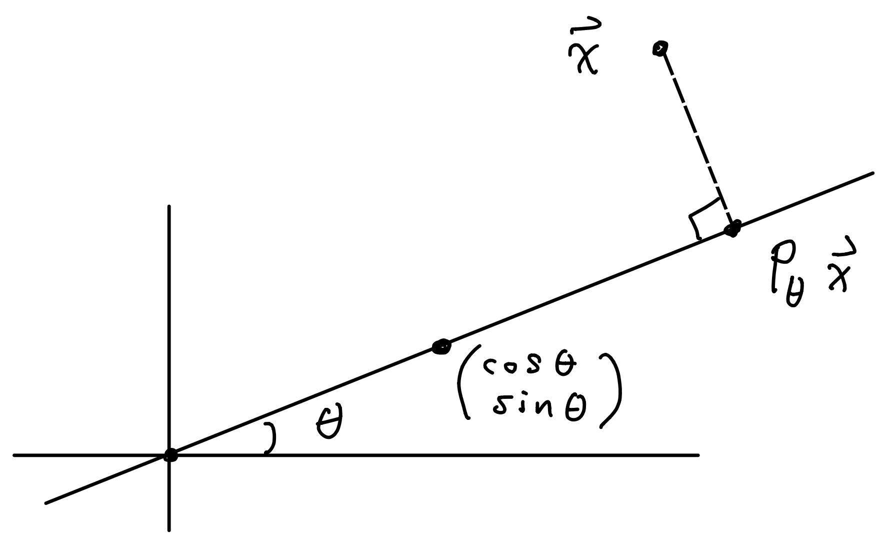
$$\boxed{\vec{a}^T\vec{a} = \vec{a}\cdot\vec{a}}$$

scalar

IF $\vec{a}\circ\vec{a} = \|\vec{a}\|^2 = 1$, then the projection matrix is just

$$P = \vec{a}\,\vec{a}^T.$$

Example: Project onto line in $\mathbb{R}^2$ with angle $\theta$ from the x-axis:



Since $\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$ has length 1:

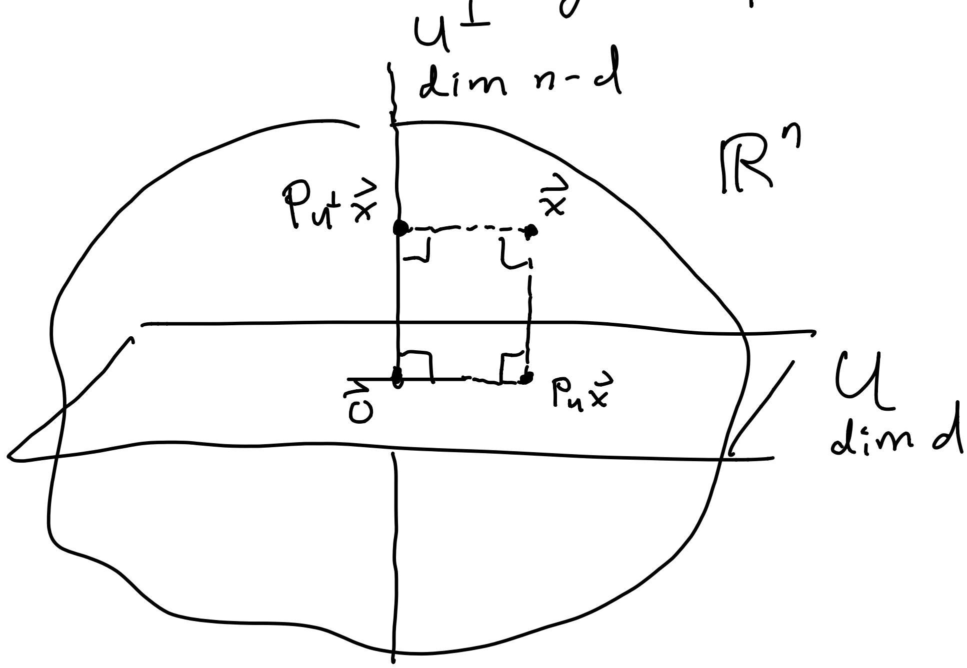
$$P_\theta = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix}$$

$$= \begin{pmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{pmatrix}.$$

see HW5.3

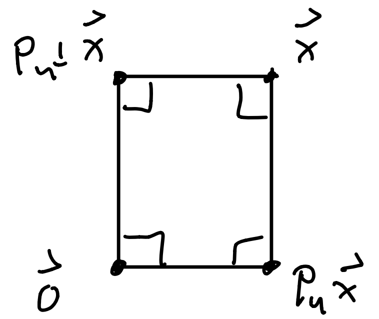
One more property of projections.

Consider complementary subspaces:



What is the relationship between the projection matrices $P_U$ & $P_{U^\perp}$ ?

For any $\vec{x}$ we have a rectangle:




$$P_U \vec{x} + P_{U^\perp}\vec{x} = \vec{x}$$

$$(P_U + P_{U^\perp})\vec{x} = \vec{x}$$

Since this is true for any $\vec{x}$, we conclude that

$$P_U + P_{U^\perp} = I \quad \text{identity matrix.}$$

This can be a useful trick.

Example: Let $U$ = plane $r\begin{pmatrix}1\\1\\1\end{pmatrix} + s\begin{pmatrix}1\\2\\3\end{pmatrix}$,

so that $U^\perp$ = line $t\begin{pmatrix}1\\-2\\1\end{pmatrix}$.

$$P_{U^\perp} = \frac{1}{\|(1,-2,1)\|^2}\begin{pmatrix}1\\-2\\1\end{pmatrix}\begin{pmatrix}1 & -2 & 1\end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix}1 & -2 & 1\\-2 & 4 & -2\\1 & -2 & 1\end{pmatrix}.$$

It follows that

$$P_u = I - P_{u^\perp}$$

$$= \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 1 & -2 & -1 \\ -2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 5 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix}.$$

Note: It is more work to use the formula

$$P_u = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}\left[\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}\right]^{-1}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}.$$

Back to Least Squares:

Find the best fit line $y = a + bx$ for data points

$$(x, y) = (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n).$$

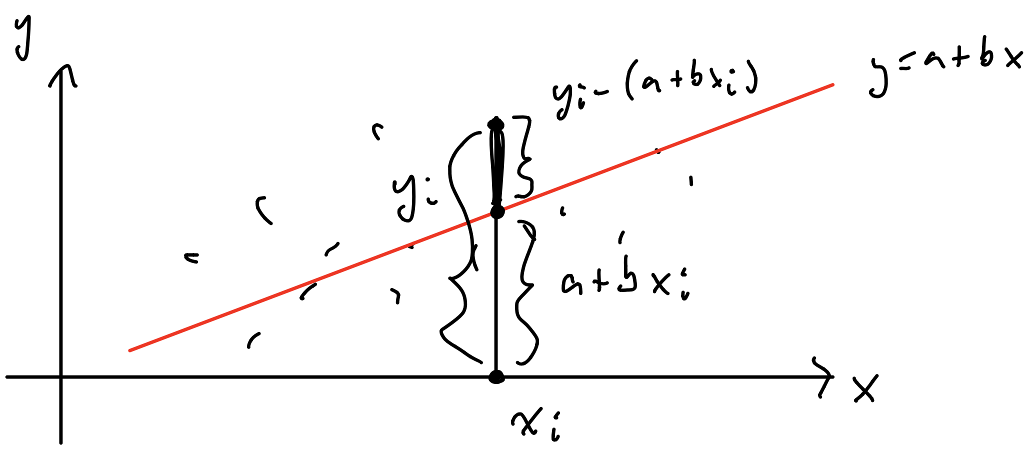
If the points were <u>on</u> the line, we would have

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$X^T \vec{a} = \vec{y}.$$

Want to minimize (squared) distance

$$\|X\vec{a} - \vec{y}\|^2 = (a + bx_1 - y_1)^2 + \dots + (a + bx_n - y_n)^2$$




So $\|X\vec{a} - \vec{y}\|^2$ = sum of the squares of the vertical errors.

Solution: The "Normal Equations"

$$X^T X \vec{a} = X^T \vec{y}.$$

$$\begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$\begin{pmatrix} n & \Sigma x \\ \Sigma x & \Sigma x^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \Sigma y \\ \Sigma xy \end{pmatrix}.$$

You may have seen this before!

We can use the same method to fit many kinds of models to data:

$y = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + b$ (hyperplane)

$y = a + bx + cx^2$ (parabola)

etc.

See HW5.1.

Major Applications of Linear Algebra:

(1) Least Squares Appoximation ✓

(2) Diagonalization ?

Now: Diagonalization.

Many applications of linear algebra involve <u>powers</u> of a square matrix.

Example: Suppose that a sequence of vectors $\vec{x}_0, \vec{x}_1, \vec{x}_2, \ldots$ follows a "linear recurrence":

$$\vec{x}_{n+1} = A\vec{x}_n ,$$

so that

$$\begin{aligned}
\vec{x}_1 &= A\vec{x}_0 \\
\vec{x}_2 &= A\vec{x}_1 = AA\vec{x}_0 = A^2\vec{x}_0 \\
\vec{x}_3 &= A\vec{x}_2 = AA^2\vec{x}_0 = A^3\vec{x}_0 \\
&\vdots \\
\vec{x}_n &= A^n\vec{x}_0 .
\end{aligned}$$

But it is hard to compute the powers of a matrix!

e.g. $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$$A^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\vdots$$

$$A^5 = \begin{pmatrix} 8 & 5 \\ 5 & 3 \end{pmatrix}$$

$$\vdots$$

$$A^{10} = \begin{pmatrix} 89 & 55 \\ 55 & 34 \end{pmatrix}$$

$$\vdots$$

$$A^n = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}$$ Is there a formula?

Yes, there is a formula. But in order to find it we first need to "diagonalize" the matrix $A$.

Definition: Given square matrix $A$, we say that $\lambda \in \mathbb{R}$ is an eigenvalue of $A$ if there exists some nonzero vector $\vec{u} \neq \vec{0}$ such that:

$$\boxed{A\vec{u} = \lambda \vec{u}}$$

In this case we say that $\vec{u}$ is an eigenvector of $A$. ///

How does this help with our analysis of the powers $A^n$?

If $A\vec{u} = \lambda\vec{u}$ then

$$\begin{aligned} A^2\vec{u} &= AA\vec{u} \\ &= A\lambda\vec{u} \\ &= \lambda A\vec{u} = \lambda\lambda\vec{u} = \lambda^2\vec{u} \end{aligned}$$

etc...

$$A^n\vec{u} = \lambda^n\vec{u}$$

power of matrix HARD

power of scalar EASY

Back to the recurrence

$$\begin{aligned}\vec{x}_n &= A\vec{x}_{n-1}\\ &\;\vdots\\ &= A^n\vec{x}_0 .\end{aligned}$$

Suppose we can express the initial condition vector $\vec{x}_0$ in terms of eigenvectors of $A$:

$$\vec{x}_0 = a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + \vec{a}_d\vec{u}_d$$

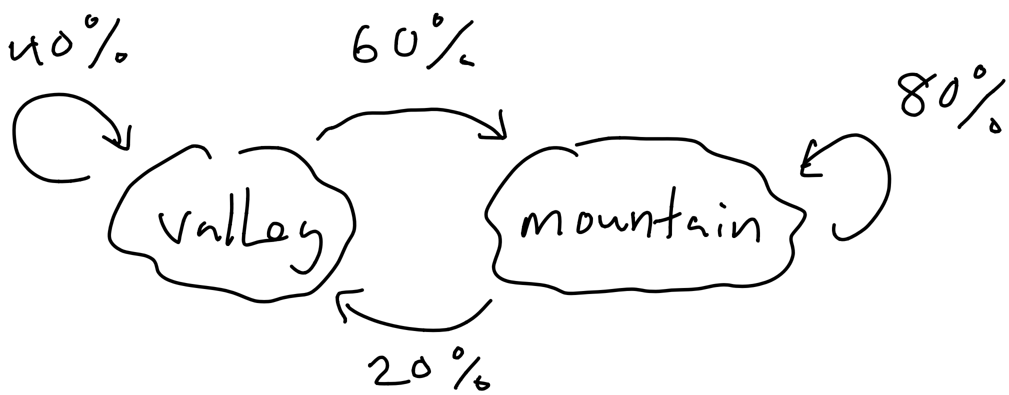
where $A\vec{u}_i = \lambda_i \vec{u}_i$ for some scalars $\lambda_i$.

Then it follows that

$$\begin{aligned}\vec{x}_n &= A^n\vec{x}_0\\ &= A^n(a_1\vec{u}_1 + \cdots + a_d\vec{u}_d)\\ &= a_1A^n\vec{u}_1 + \cdots + a_dA^n\vec{u}_d\\ &= \underbrace{a_1\lambda_1^n\vec{u}_1 + \cdots + a_d\lambda_d^n\vec{u}_d}_{\text{Everything here is easy to compute!}}\end{aligned}$$

Everything here is easy to compute! So we call this a SOLUTION.

Example: In year $n$, the are $x_n$ bears living in the valley & $y_n$ bears living on the mountain. Suppose that year-to-year migration follows the following pattern:




To be precise:

$$\begin{cases} x_{n+1} = 0.4\, x_n + 0.2\, y_n \\ y_{n+1} = 0.6\, x_n + 0.8\, y_n \end{cases}$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

If we begin with $x_0 = 100$, $y_0 = 0$, find explicit formulas for

$x_n$ & $y_n$.

$$\boxed{\text{Next Time : The Method.}}$$

Right Now : The Answer.

$$x_n = 25 + 75(0.2)^n$$
$$y_n = 75 - 75(0.2)^n.$$

Note : As $n \to \infty$, $(0.2)^n \to 0$, so that

$$x_n \to 25$$
$$y_n \to 75$$

Equilibrium :

25 bears in the valley

75 bears on the mountain.