HW4 due now(ish). (Deadline extended until noon 12pm.)

Today: HW4 Discussion
Quiz 4 Topics
Preview of next week (?)

HW4 Discussion.

Problem 1: SHAPES of Matrices.

Most important property of matrix multiplication is "associativity":

$$A(BC) = (AB)C.$$

Why does this hold? Suppose

A has shape $k \times l$
B has shape $l \times m$
C has shape $m \times n$

so... AB has shape $k \times m$
BC has shape $l \times n$.

We should think of these as

Linear functions:

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "C"] \arrow[rr, bend left, "BC"] \arrow[rrr, bend right=60] & \mathbb{R}^m \arrow[r, "B"] \arrow[rr, bend right, "AB"'] & \mathbb{R}^l \arrow[r, "A"] & \mathbb{R}^k
\end{tikzcd}$$

$$A(BC) = (AB)C = ABC$$

Composition of functions is naturally associative!

Example:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \vec{z} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$A(B\vec{z}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

$$(AB)\vec{z} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

SAME ✓

Now let $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Compute $\vec{z}^T A \vec{x}$

$$= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$= \begin{pmatrix} 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ <u>OR</u> $\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

$= 6$ <u>OR</u> 6 (same ✓)

[Remark: To me, a 1×1 matrix is a scalar, so I don't write parentheses around it.]

Finally, let $\vec{y} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and compute

$$
\begin{aligned}
& BA + \vec{x}\,\vec{y}^{T} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}
\end{aligned}
$$

Inversion: Say $m \times n$ matrix A is invertible if there exists some $n \times m$ matrix B such that

$$\underset{m\times n}{A}\,\underset{n\times m}{B} = \underset{m\times m}{I} \quad \text{identity matrix}$$

$$\underset{n\times m}{B}\,\underset{m\times n}{A} = \underset{n\times n}{I} \quad \text{identity matrix.}$$

Theorem: If $m \neq n$ then this is impossible!

If $m = n$, then such B exists if and only if $\text{rank}(A) = m = n$, i.e. the rows & columns of A are independent.

More succinctly,

$$\underset{n\times n}{A} \text{ is invertible} \iff RREF(A) = \underset{n\times n}{I}$$

In this case the inverse is unique, we call it A^{-1}, and we can find it with the following trick:

$$(A \mid I) \xrightarrow[RREF]{} (I \mid A^{-1}).$$

Problem 3: $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 2 & 4 \end{pmatrix}$.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & 0 & -3 \\ 0 & 1 & 0 & 4 & -1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 2 & 1 \\ 0 & 1 & 0 & 4 & -1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

Conclusion: $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 2 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} -4 & 2 & 1 \\ 4 & -1 & -2 \\ -1 & 0 & 1 \end{pmatrix}$

We can use this to solve any linear system of the form

$$\begin{cases} x + 2y + 3z = a \\ 2x + 3y + 4z = b \\ x + 2y + 4z = c \end{cases}, \text{ i.e., } A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Solution: Multiply on the left by A^{-1}.

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 & 2 & 1 \\ 4 & -1 & -2 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4a + 2b + c \\ 4a - b - 2c \\ -a + c \end{pmatrix}$$

We just solved an infinite family of linear systems.

Problem 4c,d: The power of matrix arithmetic.

c) If A^{-1} exists then $(A^T)^{-1}$ exists.

Proof: I claim that $(A^T)^{-1} = (A^{-1})^T$.

Check: $A^T (A^{-1})^T = I$?

$(A^{-1})^T A^T = I$?

[Recall: $(CD)^T = D^T C^T$.]

So: Put $D = A$ & $C = A^{-1}$ to get

$A^T (A^{-1})^T = (A^{-1} A)^T = I^T = I$ ✓

Put $D = A^{-1}$ & $C = A$ to get

$(A^{-1})^T A^T = (A A^{-1})^T = I^T = I$ ✓

d) If A^{-1}, B^{-1}, AB exist then $(AB)^{-1}$ exists.

Proof: I claim that $(AB)^{-1} = B^{-1} A^{-1}$, which exists because B^{-1} & A^{-1} are square matrices of the same shape.

Need to show: $(AB)(B^{-1}A^{-1}) = I$,

$(B^{-1}A^{-1})(AB) = I$.

Check: $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$= A I A^{-1}$

$$= AA^{-1}$$
$$= I \quad \checkmark$$

$$
\begin{aligned}
(B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B \\
&= B^{-1}IB \\
&= B^{-1}B \\
&= I \quad \checkmark
\end{aligned}
$$

More Conceptual Proof:

$$
\begin{tikzcd}
\mathbb{R}^n \arrow[r, shift left, "B"] \arrow[rr, bend left=50, "AB"] & \mathbb{R}^n \arrow[l, shift left, "B^{-1}"] \arrow[r, shift left, "A"] & \mathbb{R}^n \arrow[l, shift left, "A^{-1}"] \arrow[ll, bend left=50, "B^{-1}A^{-1}"]
\end{tikzcd}
$$

Some <u>special</u> 2×2 matrices:

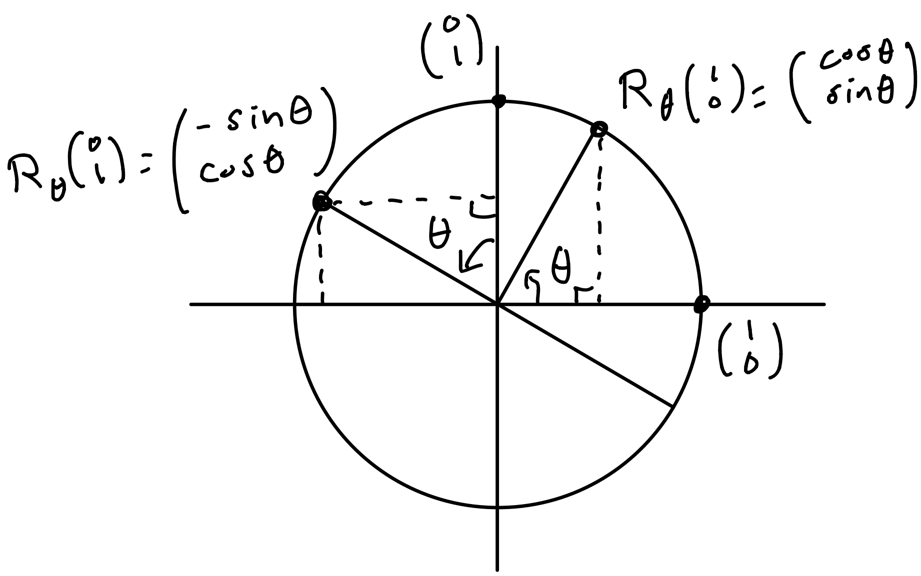
For any angle θ, we define

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \& \quad F_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

These matrices have some very interesting properties.

Claim: $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ rotates every point c.c.w. by θ around the origin.

Since rotation is <u>linear</u>, we only need to check that this works on a <u>basis</u>.



✓

From this we see that

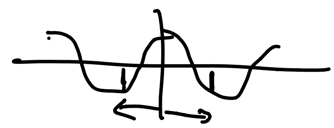
$$R_\theta^{-1} = \text{rotate \underline{clockwise} by } \theta.$$

$$= \text{rotate c.c.w. by } -\theta.$$

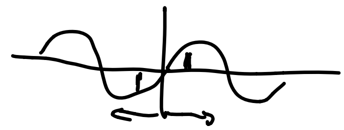
$$= R_{-\theta}$$

$$= \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$



$$= R_\theta^T$$



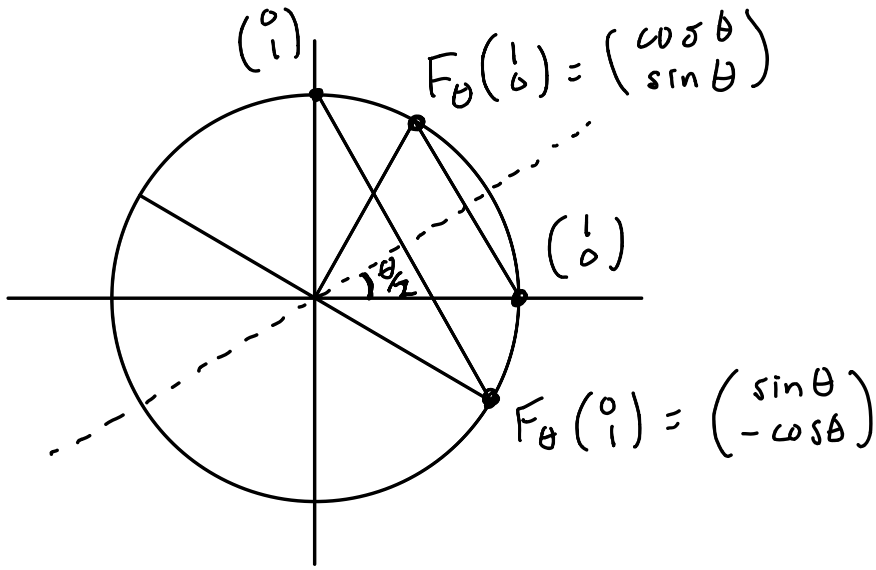
Conclusion: $R_\theta^{-1} = R_\theta^T$.

[Jargon: Any matrix satisfying

$$A^{-1} = A^T$$

is called an <u>orthogonal matrix</u>.
Equivalently, the rows (or columns) of A are an <u>orthonormal basis</u>, i.e., are perpendicular of length 1.]

What about F_θ?



This is actually a reflection across the line that has angle $\theta/2$ from the x-axis. From this we see that

$$\begin{aligned} F_\theta^{-1} &= \text{undo the reflection} \\ &= \text{do the same reflection again} \\ &= F_\theta. \end{aligned}$$

Also observe that

$$F_\theta^T = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}^T = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = F_\theta.$$

Hence $F_\theta^{-1} = F_\theta = F_\theta^T$.

so F_θ is another example of an orthogonal matrix. In fact, there is a fancy theorem that says that <u>any</u> orthogonal matrix $A^{-1} = A^T$ (of any size) is a composition of

reflections & rotations.

Topics for Quiz 4:

- Definition & Computation of Matrix Multiplication.

(ij entry of AB) = (ith row A)(jth col B)

(jth col AB) = A(jth col B)

(ith row AB) = (ith row A) B.

- Definition & Computation of Matrix Inverses.
- Rules of Matrix Arithmetic.