HW 4 due tomorrow <u>before</u> class.

Today: Invertibility of Matrices.

A function $f: X \to Y$ between sets is *invertible* if there exists a function $g: Y \to X$ in the opposite direction such that

$g \circ f : X \to X$ is the identity function

and

$f \circ g : Y \to Y$ is the identity function.

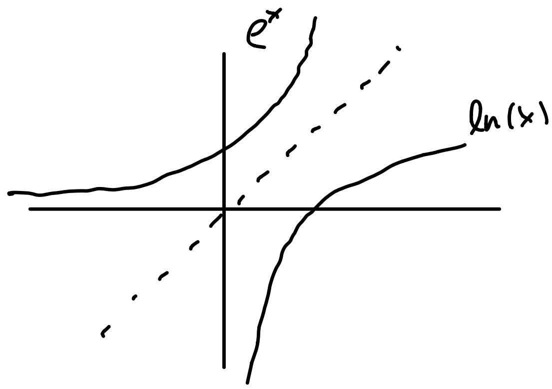
In other words, we require

$$g(f(x)) = x \text{ for all } x \in X$$
$$f(g(y)) = y \text{ for all } y \in Y.$$

In this case the function g is <u>unique</u> and we write

$$g = \text{"}f^{-1}\text{"}$$
$$= \text{"}\underline{\text{the}} \text{ inverse of } f\text{"}$$

Example : $f(x) = e^x$, $f: \mathbb{R} \to \mathbb{R}_{>0}$.
$g(x) = \ln(x)$, $g: \mathbb{R}_{>0} \to \mathbb{R}$.



Non Example : $\sin : \mathbb{R} \to [-1, 1]$
is <u>not</u> invertible, but $\sin : [0, 2\pi) \to [-1, 1]$
<u>is</u> invertible : $\sin^{-1} : [-1, 1] \to [0, 2\pi)$.

Theorem : $f : X \to Y$ is invertible
if and only if :

- it is "one-to-one", meaning that
  $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$.
- it is "onto", meaning that for
  any $y \in Y$ there exists some $x \in X$
  such that $f(x) = y$.

Special Case: Let A be $m \times n$ matrix and consider the linear function

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

① When is A "one-to-one"?

② When is A "onto"?

$\boxed{\text{suppose } r = \text{rank}(A)}$

Theorem:

① A is "one-to-one"

$\iff N(A) = \{\vec{0}\}$

$\iff \dim N(A) = 0$

$\iff n - r = 0$

$\iff r = n$

$\iff$ the columns of A are independent.

$\iff$ there exists at least one matrix C such that

$$\underset{n\times m}{C}\,\underset{m\times n}{A} = \underset{n\times n}{I}.$$

② A is "onto"

$\iff C(A) = \mathbb{R}^m$

$\iff \dim C(A) = m$

$\iff r = m.$

$\iff$ the <u>rows</u> of A are independent

$\iff$ there exists at least one matrix B such that

$$\underset{m\times n}{A}\;\underset{n\times m}{B} = \underset{m\times m}{I}$$

///

Example: $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \end{pmatrix}$. $(r=2)$

Since rank = # rows, we know that A has at least one right inverse.

$$AB = I$$

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad\Bigg|\quad \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1+2c \\ 3c \\ c \end{pmatrix} \qquad \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 2f \\ 1+3f \\ f \end{pmatrix}$$

$$\Longrightarrow \quad B = \begin{pmatrix} 1+2c & 2f \\ 3c & 1+3f \\ c & f \end{pmatrix}$$

Infinitely many solutions!

e.g., take $c = f = 0$:

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Next, since rank $\neq$ # columns, I claim that A has <u>NO</u> left inverse.

<u>Proof</u>: The columns of $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \end{pmatrix}$ are <u>not</u> independent. To be specific, we have a nontrivial relation:

$$2(\text{col } 1) + 3(\text{col } 2) + 1(\text{col } 3) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

In matrix language, this tells us a nonzero vector in the nullspace:

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A\vec{x} = \vec{0}.$$

<u>If</u> we had a left inverse, say $\underset{3\times 2}{C}\,\underset{2\times 3}{A} = \underset{3\times 3}{I}$, <u>then</u> this would imply

$$A\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$CA\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = C\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$I\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \underset{3\times 2}{C}\,\underset{2\times 1}{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$

$$\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{NOPE!}$$

[Remark: $\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.]

HW4.4b: If $A\vec{x} = \vec{0}$ and $\vec{x} \neq \vec{0}$ then A has no left inverse (hence also no two-sided inverse). Proof: If we had $CA = I$ then we would have

$$A\vec{x} = \vec{0}$$
$$CA\vec{x} = C\vec{0}$$
$$I\vec{x} = C\vec{0}$$
$$\vec{x} = \vec{0}$$

Contradiction!

Two-Sided Inverse?

Theorem: If $AB = I$ & $CA = I$ then we must have

$$B = IB = (CA)B = C(AB) = CI = C$$

In other words, if A has both a left and a right inverse, then it has a unique <u>two-sided inverse</u>.

Call it A^{-1}.

(1) When does it exist?
(2) How can we compute it?

Theorem:

(1) A^{-1} exists

$\iff$ A is one-to-one & onto

$\iff$ $r = n$ & $r = m$ (hence $m = n$)

In other words, A^{-1} exists when

- A is <u>square</u>, $(m = n)$
- Rows of A are independent, $(r = m)$
- Columns of A are independent. $(r = n)$

(2) To compute A^{-1} there is a GOOD TRICK. Form the augmented matrix $(A \mid I)$. Compute RREF:

$$(A \mid I) \underset{\text{RREF}}{\rightsquigarrow} (I \mid A^{-1}).$$

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Example: $A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$.

SQUARE ✓

RANK = # ROWS = # COLS ✓

Compute the inverse $A^{-1} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 2 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & -2 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right)$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 2 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -2 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & -1/2 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1/2 \end{array}\right)$$

$$\begin{cases} a + 0b = -1 \\ 0a + b = 1 \end{cases} \qquad \begin{cases} c + 0d = 1 \\ 0c + d = -1/2 \end{cases}$$

$$\Longrightarrow A^{-1} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1/2 \end{pmatrix}.$$

[Remark: And the solution is unique!]

Check:

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

Do we need to check the other direction? $\begin{pmatrix} -1 & 1 \\ 1 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} = ?$

NO. This is guaranteed to work out.

Theorem: If A, B are square then

$$AB = I \iff BA = I.$$

You only need to check one of these.

Proof: This is quite subtle, but it follows from the above theorems. ///

Note that to find $\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}^{-1}$

we had to compute the RREF

of two <u>very similar</u> matrices:

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 2 & 0 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 2 & 1 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1/2 \end{array}\right)$$

The GOOD TRICK just tells us to perform these computations <u>simultaneously</u>:

$$\left(\begin{array}{cc|c|c} 1 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1/2 \end{array}\right)$$

If remove one of the vertical bars then this becomes

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1/2 \end{array}\right)$$

$$(A \mid I) \rightsquigarrow (I \mid A^{-1}) \quad \text{NICE} \; \smile$$

Another Example:

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 3 & 4 \end{pmatrix}, \quad A^{-1} = ?$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right)$$

oops!

What Happened?

We just found out that

$$\operatorname{rank}(A) = 2.$$

So the inverse does not exist!

Hint for HW 4.3:

If A^{-1} exists, then the matrix equation $A\vec{x}=\vec{b}$ can be solved symbolically:

$$A\vec{x} = \vec{b}$$
$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$
$$I\vec{x} = A^{-1}\vec{b}$$
$$\vec{x} = A^{-1}\vec{b}.$$

This is the power of matrix arithmetic. It can turn complicated problems into simple algebra.

Preview for next week:

Suppose that the equation $A\vec{x}=\vec{b}$ has <u>no solution</u>. What can we do?

The "Least Squares Approximation"

tells us to multiply on the left by A^T to get

$$A^T A \vec{x} = A^T \vec{b}.$$

Under good conditions, the matrix $A^T A$ <u>is</u> invertible, hence

$$\vec{x} = (A^T A)^{-1} A^T \vec{b}.$$

This is not an exact solution to the original $A\vec{x} = \vec{b}$ (because it has no exact solution) but it is a "best approximate solution" in some precise sense that we will discuss next week.