June 22 - June 28

We have finished discussing "least squares regression", which is one of the most common applications of Linear Algebra.

There is one more application of Linear Algebra I want to discuss before sending you out into the world. I'll call it

"spectral analysis"

and I'll also introduce this topic with an example.

Motivating Example: You may have heard of the "Fibonacci Sequence"

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, etc.

If we write f_n for the n^{th} Fibonacci number then the sequence is defined by the "initial conditions"

$$f_0 = 0 \quad \& \quad f_1 = 1$$

and the "recurrence equation"

$$f_{n+2} = f_{n+1} + f_n \quad \text{for all } n \geq 0.$$

For example, we have

$$\begin{aligned} f_2 &= f_1 + f_0 = 1 + 0 = 1 \\ f_3 &= f_2 + f_1 = 1 + 1 = 2 \\ f_4 &= f_3 + f_2 = 2 + 1 = 3 \\ f_5 &= f_4 + f_3 = 3 + 2 = 5, \text{ etc.} \end{aligned}$$

Our goal today is to find a "closed formula for the n^{th} Fibonacci number:

$$f_n = ?$$

The answer is very hard to guess, but we can compute it rather easily using a trick and some Linear Algebra. The trick is to rewrite the recurrence equation as a system of two linear equations

$$\begin{cases} f_{n+2} = f_{n+1} + f_n \\ f_{n+1} = f_{n+1} \end{cases}$$

The second equation looks quite useless but it's not because it allows us to express the recurrence as a matrix equation

$$(*) \qquad \begin{pmatrix} f_{n+2} \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}.$$

To save some space we will introduce the notations

$$\vec{f}_n = \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} \quad \& \quad T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

Then we can express the initial conditions and the recurrence as follows:

- $\vec{f}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- $\vec{f}_{n+1} = T\vec{f}_n$ for all $n \geqslant 0$.

Now we're ready to apply Linear Algebra. By computing the first few vectors $\vec{f}_n$,

$$\vec{f}_1 = T\vec{f}_0$$

$$\vec{f}_2 = T\vec{f}_1 = T(T\vec{f}_0) = (TT)\vec{f}_0 = T^2\vec{f}_0$$

$$\vec{f}_3 = T\vec{f}_2 = T(T^2\vec{f}_1) = (TT^2)\vec{f}_0 = T^3\vec{f}_0$$

we see that the n^{th} vector is given by

$$\vec{f}_n = T^n\vec{f}_0$$

$$\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

and we really only care about the 2nd entry of this vector, which is the n^{th} Fibonacci number f_n.

Great, now let's compute it. HOW?

This is where the "spectral analysis" comes in. We make the following fundamental definition.

☆ Definition: We will say that a number λ is an <u>eigenvalue</u> for the matrix T if there exists a nonzero vector $\vec{x} \neq \vec{0}$ such that

$$\boxed{T\vec{x} = \lambda\vec{x}}$$

and in this case we will say that $\vec{x}$ is a <u>λ-eigenvector</u> of T. ///

The whole reason for this definition is the following observation: If $\vec{x}$ is a λ-eigenvector for T then we have

$$\begin{aligned} T^2\vec{x} &= T(T\vec{x}) \\ &= T(\lambda\vec{x}) \\ &= \lambda(T\vec{x}) \\ &= \lambda(\lambda\vec{x}) = \lambda^2\vec{x}, \end{aligned}$$

$$
\begin{aligned}
T^3\vec{x} &= T(T^2\vec{x}) \\
&= T(\lambda^2\vec{x}) \\
&= \lambda^2(T\vec{x}) \\
&= \lambda^2(\lambda\vec{x}) = \lambda^3\vec{x},
\end{aligned}
$$

and in general we have

$$\boxed{T^n\vec{x} = \lambda^n\vec{x}}.$$

So if $\vec{f}_0 = (1,0)$ were an eigenvector of T we would be done. Unfortunately it's not:

$$T\vec{f}_0 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

which is not of the form $\lambda\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. But that's okay. Here's the big idea.

☆ Idea of Spectral Analysis:

If we can express the initial condition $\vec{f}_0$ as a linear combination of eigenvectors for the transition matrix T, then we will be done.

Indeed, suppose that $\vec{u}$ & $\vec{v}$ are eigenvectors for T with

$$T\vec{u} = \lambda \vec{u} \quad \& \quad T\vec{v} = \mu \vec{v},$$

for some eigenvalues λ & μ and suppose that we can write

$$\vec{f_0} = a\vec{u} + b\vec{v}$$

for some numbers a & b. Then we will have

$$\begin{aligned} T^n \vec{f_0} &= T^n(a\vec{u} + b\vec{v}) \\ &= a(T^n\vec{u}) + b(T^n\vec{v}) \\ &= a\lambda^n \vec{u} + b\mu^n \vec{v} \end{aligned}$$

and the problem will be solved! Thus we have reduced the problem to:

- findining enough eigenvectors for T
- expressing the initial condition $\vec{f_0}$ in terms of them. ///

Right now I am introducing the idea of "spectral analysis" through a motivational example.

Recall the Fibonacci numbers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots .$$

These are defined by initial conditions

$$f_0 = 0 \quad \& \quad f_1 = 1 ,$$

and by the recurrence equation

$$f_{n+2} = f_{n+1} + f_n \quad \text{for } n \geq 0 .$$

Our goal is to "solve" this recurrence, i.e., to find a "closed formula" for the n^{th} Fibonacci number. The answer is very hard to guess so it is preferable to develop a mechanical technique.

To do this we will define the vectors

$$\vec{f}_n := \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}$$

consisting of two consecutive Fibonacci numbers and then observe that the initial conditions and recurrence can be rewritten in terms of matrix algebra as

$$\vec{f}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \& \quad \vec{f}_{n+1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \vec{f}_0 \text{ for } n \geq 0.$$

If we define $T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ then we can solve explicitly for $\vec{f}_n$:

$$\boxed{\vec{f}_n = T^n \vec{f}_0}$$

Now the whole problem is to investigate the powers T^n of the matrix T, and the key tools for doing this are called eigenvalues & eigenvectors;

☆ Consider a nonzero vector $\vec{x} \neq \vec{0}$. We say that $\vec{x}$ is an eigenvector for the matrix T if there exists some number λ such that

$$T\vec{x} = \lambda \vec{x}.$$

In this case we say that λ is the eigenvalue corresponding to the eigenvector $\vec{x}$. (Sometimes we say that $\vec{x}$ is a "λ-eigenvector" of T.) ///

☆ The idea of spectral analysis is to express the initial condition $\vec{f}_0$ as a linear combination of eigenvectors for the transition matrix.

I'll show you how to compute the eigenvectors in a bit. Right now let me just tell you the answer.

If we define the numbers

$$\varphi_1 = \frac{1+\sqrt{5}}{2} \quad \& \quad \varphi_2 = \frac{1-\sqrt{5}}{2}$$

then I claim [just believe me] that.

- $$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ 1 \end{pmatrix} = \varphi_1 \begin{pmatrix} \varphi_1 \\ 1 \end{pmatrix}$$

- $$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varphi_2 \\ 1 \end{pmatrix} = \varphi_2 \begin{pmatrix} \varphi_2 \\ 1 \end{pmatrix}$$

- $$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi_1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi_2 \\ 1 \end{pmatrix}$$

And then the answer to our problem is immediate. We have

$$\begin{aligned} \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} &= \vec{f}_n = T^n \vec{f}_0 \\ &= T^n \left(\frac{1}{\sqrt{5}} \begin{pmatrix} \varphi_1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi_2 \\ 1 \end{pmatrix} \right) \\ &= \frac{1}{\sqrt{5}} T^n \begin{pmatrix} \varphi_1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}} T^n \begin{pmatrix} \varphi_2 \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{5}} \varphi_1^n \begin{pmatrix} \varphi_1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}} \varphi_2^n \begin{pmatrix} \varphi_2 \\ 1 \end{pmatrix}. \end{aligned}$$

Then comparing the second entry in each vector gives us the desired formula for the n^{th} Fibonacci number:

$$f_n = \frac{1}{\sqrt{5}} \varphi_1^n - \frac{1}{\sqrt{5}} \varphi_2^n$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \quad (!)$$

I consider this formula pretty amazing because it doesn't even look like a whole number. Let's check a couple of cases:

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^0 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^0$$

$$= \frac{1}{\sqrt{5}} \cdot 1 - \frac{1}{\sqrt{5}} \cdot 1 = 0 = f_0 \quad \checkmark$$

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^1 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^1$$

$$= \frac{1}{2\sqrt{5}}\left[(\cancel{1}+\sqrt{5}) - (\cancel{1}-\sqrt{5}) \right]$$

$$= \frac{1}{\cancel{2\sqrt{5}}}\left[\cancel{2\sqrt{5}}\right] = 1 = f_1 \quad \checkmark.$$

OK, that's good enough for me :) .

What remains to do?

I need to show you how to compute the eigenvalues & eigenvectors of a matrix if you don't know them already. Actually, this is pretty hard in general so I'll just show you how to do it for 2×2 matrices.

So let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and suppose that λ is eigenvalue of A. This means that there exists a vector $\vec{x} \neq \vec{0}$ such that

$$
\begin{aligned}
A\vec{x} &= \lambda \vec{x} \\
A\vec{x} &= \lambda I_2 \vec{x} \\
A\vec{x} - \lambda I_2 \vec{x} &= \vec{0} \\
(A - \lambda I_2)\vec{x} &= \vec{0}.
\end{aligned}
$$

Since $\vec{x} \neq \vec{0}$ this equation tells me that the matrix $A - \lambda I_2$ has a non-trivial column relation, so it is not invertible.

If $A - \lambda I_2$ <u>were</u> invertible then its inverse would be given by the formula

$$(A - \lambda I_2)^{-1} = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]^{-1}$$

$$= \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}^{-1}$$

$$= \frac{1}{(a-\lambda)(d-\lambda) - bc} \begin{pmatrix} d-\lambda & -b \\ -c & a-\lambda \end{pmatrix}.$$

But since we know that $A - \lambda I_2$ is <u>not</u> invertible, it must be the case that

$$(a-\lambda)(d-\lambda) - bc = 0$$
$$ad - a\lambda - d\lambda + \lambda^2 - bc = 0$$

$$\boxed{\lambda^2 - (a+d)\lambda + (ad - bc) = 0}$$

This is called the characteristic equation of the matrix A. Its solutions λ are precisely the eigenvalues of A, and we can compute them using the quadratic formula:

(*) $$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

After finding the eigenvalues from (*) it is easy to find the corresponding eigenvectors by solving the linear system

$$(A - \lambda I_2)\vec{x} = \vec{0}$$

for each eigenvalue λ.

Now: E.values & E.vectors

Consider the matrix $A = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}$.

Using a computer we find

$$A^2 = \begin{pmatrix} .70 & .45 \\ .30 & .55 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} .650 & .525 \\ .350 & .475 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} .6250 & 0.5622 \\ .3750 & 0.4375 \end{pmatrix}$$

$\vdots$

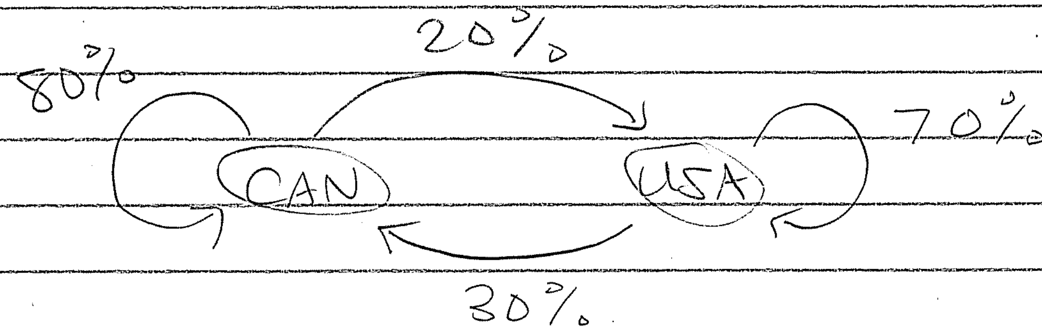
$$A^{10} = \begin{pmatrix} 0.600 & 0.599 \\ 0.399 & 0.401 \end{pmatrix}$$

It seems like

$$\lim_{n\to\infty} A^n = \begin{pmatrix} .6 & .6 \\ .4 & .4 \end{pmatrix}$$

What's going on here?

Consider a simple model. A species of bird lives in Canada and the USA. Every year there is a migration



Assume no birds are born or die.
In year n there are

c_n birds in CAN.
u_n birds in USA

How are $\begin{pmatrix} c_n \\ u_n \end{pmatrix}$ and $\begin{pmatrix} c_{n+1} \\ u_{n+1} \end{pmatrix}$ related?

Of the c_n birds in CAN now, $.8c_n$ stay and $.2c_n$ move. Of the u_n birds in USA now, $.7u_n$ stay and $.3u_n$ move.

Hence

$$c_{n+1} = .8c_n + .3u_n$$
$$u_{n+1} = .2u_n + .7u_n$$

i.e. $$\begin{pmatrix} c_{n+1} \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix} \begin{pmatrix} c_n \\ u_n \end{pmatrix}$$

$$\vec{v}_{n+1} = A\vec{v}_n$$

Say $\vec{v}_n$ is the state vector at time n

Say A is the transition matrix.

Example

Start with $\vec{v}_0 = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$

Then $\vec{v}_1 = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix} \begin{pmatrix} 10 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$

$$\vec{v}_2 = A\vec{v}_1 = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix} \begin{pmatrix} 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

$$\vec{v}_3 = A\vec{v}_2 = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix} \begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 6.5 \\ 3.5 \end{pmatrix}$$

Q: 6.5 birds?
A: Yes. We're just dealing with probabilities.

In general we have

$$\begin{aligned} \vec{v}_n &= A\vec{v}_{n-1} \\ &= AA\vec{v}_{n-2} \\ &= AAA\vec{v}_{n-3} \\ &\vdots \\ &= \underbrace{AAA \cdots A}_{n \text{ times}} \vec{v}_0 \\ &= A^n \vec{v}_0 = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}^n \begin{pmatrix} 10 \\ 0 \end{pmatrix} \end{aligned}$$

Can we compute this?

Big Idea: We will say state $\vec{v}$ is an equilibrium of the system if

$$A\vec{v} = \vec{v} = 1\vec{v}.$$

An eigenvector with eigenvalue 1

If it exists, let's compute it.

Let $\vec{v} = \begin{pmatrix} c \\ u \end{pmatrix}$.

Then $A\vec{v} = \vec{v}$.

$$\Rightarrow \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}\begin{pmatrix} c \\ u \end{pmatrix} = \begin{pmatrix} c \\ u \end{pmatrix}.$$

$$\Rightarrow \begin{aligned} .8c + .3u &= c \\ .2c + .7u &= u. \end{aligned}$$

$$\Rightarrow \begin{aligned} -.2c + .3u &= 0 \\ \cancel{.2c - .3u = 0}. \end{aligned}$$ redundant. :)

$$\Rightarrow \begin{aligned} .3u &= .2c \\ 3u &= 2c. \end{aligned}$$

$$\Rightarrow u/c = 2/3$$

The 1-eigenspace of A is the line

$$\begin{pmatrix} c \\ u \end{pmatrix} = t \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

In particular we have

$$A \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}.$$

6 birds in CAN
4 birds in USA is an equilibrium.

But we haven't yet explained why

$$A^n \begin{pmatrix} 10 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

as $n \longrightarrow \infty$

To do this we need the <u>other</u> eigenvalue.

The characteristic equation of $\begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}$ is

$$(.8-\lambda)(.7-\lambda) - (.2)(.3) = 0$$

$$.56 - .8\lambda - .7\lambda + \lambda^2 - .06 = 0$$

$$\lambda^2 - 1.5\lambda + .5 = 0$$

$$2\lambda^2 - 3\lambda + 1 = 0$$

Hence the eigenvalues are

$$\lambda = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(2)}}{2(2)} = \frac{3 \pm 1}{4}$$

$$= 1 \text{ or } .5$$

Let's compute the eigenvalues corresponding to eigenvalue .5

$$\begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}\begin{pmatrix} c \\ u \end{pmatrix} = .5\begin{pmatrix} c \\ u \end{pmatrix}$$

$$\Longrightarrow \begin{aligned} .8c + .3u &= .5c \\ .2c + .7u &= .5u \end{aligned}$$

$\Longrightarrow \quad .3c + .3u = 0$

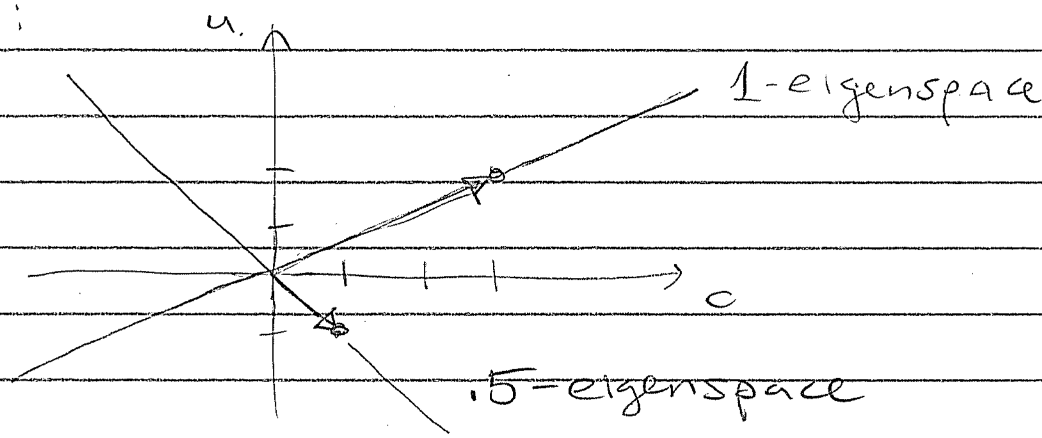
~~.2c + .2u = 0~~ Redundant ☺

$\Longrightarrow \quad c + u = 0.$

So the ".5-eigenspace" is the line

$$\begin{pmatrix} c \\ u \end{pmatrix} = t \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Picture:



Slogan: Once you know the eigenvectors, you should express everything in terms of them.

For example, let's express our initial state vector:

$$\begin{pmatrix} 10 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 2 \end{pmatrix} + 4\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Then we have

$$A^n \begin{pmatrix} 10 \\ 0 \end{pmatrix} = A^n \left[2\begin{pmatrix} 3 \\ 2 \end{pmatrix} + 4\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right]$$

$$= 2A^n \begin{pmatrix} 3 \\ 2 \end{pmatrix} + 4A^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= 2\begin{pmatrix} 3 \\ 2 \end{pmatrix} + 4\left(\frac{1}{2}\right)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 + 4/2^n \\ 4 - 4/2^n \end{pmatrix}$$

As $n \to \infty$ we have

$$A^n \begin{pmatrix} 10 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 6 + 0 \\ 4 + 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

Today: HW 8 Discussion

6.1.9. Assume that x is an e.vector of A with e.value λ.

(a) Then x is also an e.vector of A^2, but with e.value λ^2.

Proof: We have

$$A^2\vec{x} = A(A\vec{x}) = A(\lambda\vec{x})$$
$$= \lambda A\vec{x} = \lambda\lambda\vec{x} = = \lambda^2\vec{x}$$

(b) Then x is also an e.vector of A^{-1} (if A^{-1} exists), but with e.value λ^{-1} $\left(= \frac{1}{\lambda}\right)$.

Proof: We have

$$\vec{x} = I\vec{x} = (A^{-1}A)\vec{x} = A^{-1}(A\vec{x})$$
$$= A^{-1}(\lambda\vec{x})$$
$$= \lambda(A^{-1}\vec{x}).$$

Hence

$$A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}.$$ $\blacksquare$

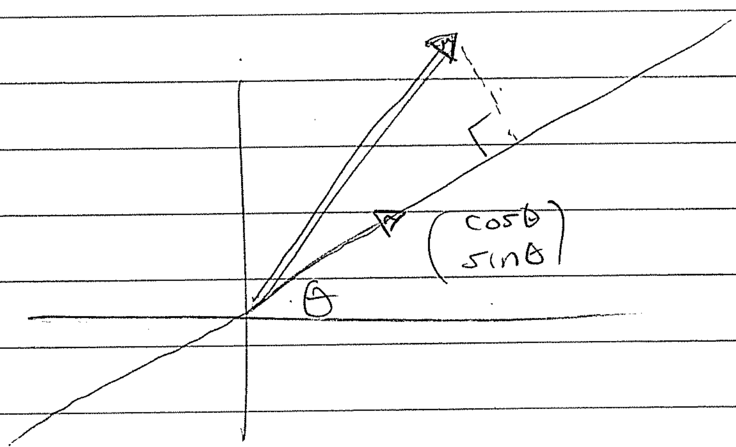
(c) Then x is also an e.vector of $A+I$ but with e.value $\lambda+1$.

Proof: We have

$$\begin{aligned}(A+I)\vec{x} &= A\vec{x} + I\vec{x} \\ &= \lambda\vec{x} + 1\vec{x} \\ &= (\lambda+1)\vec{x}\end{aligned}$$ $\blacksquare$

Problem A.1.

Let P be the matrix that projects onto the line through $(\cos\theta, \sin\theta)$.



To save space, let's write $c = \cos\theta$ and
$s = \sin\theta$.

The projection matrix is $P = \dfrac{\vec{a}\,\vec{a}^T}{\vec{a}^T\vec{a}}$ where $\vec{a} = \begin{pmatrix} c \\ s \end{pmatrix}$

$$\Rightarrow \quad P = \frac{\begin{pmatrix} c \\ s \end{pmatrix}\begin{pmatrix} c & s \end{pmatrix}}{\begin{pmatrix} c & s \end{pmatrix}\begin{pmatrix} c \\ s \end{pmatrix}} = \frac{1}{c^2+s^2}\begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}$$

$$= \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}$$

because $c^2 + s^2 = 1$, as you know.

The e.values of P are given by

$$(c^2 - \lambda)(s^2 - \lambda) - (cs)(cs) = 0$$

$$\cancel{c^2 s^2} - c^2\lambda - s^2\lambda + \lambda^2 - \cancel{c^2 s^2} = 0$$

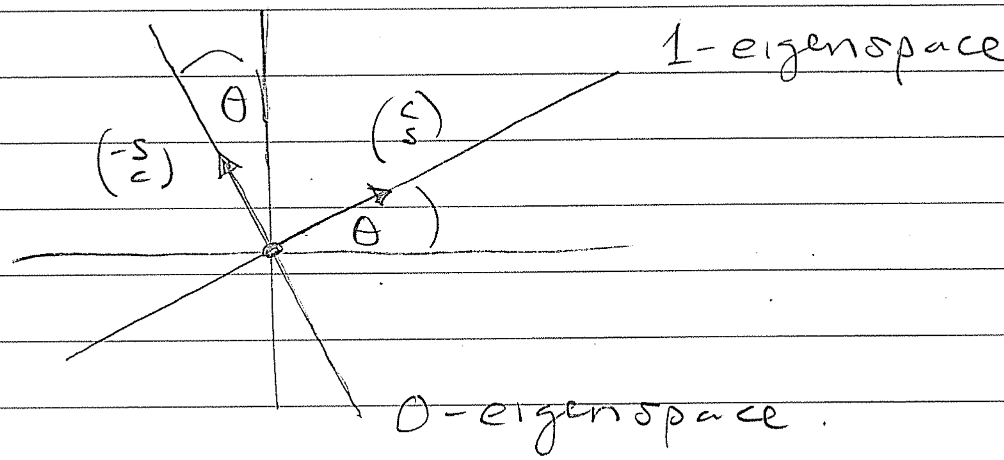
$$\lambda^2 - (c^2 + s^2)\lambda = 0.$$

$$\lambda^2 - \lambda = 0$$

$$\lambda(\lambda - 1) = 0$$

$\Rightarrow$ E.values are $\lambda = 1$ and 0.

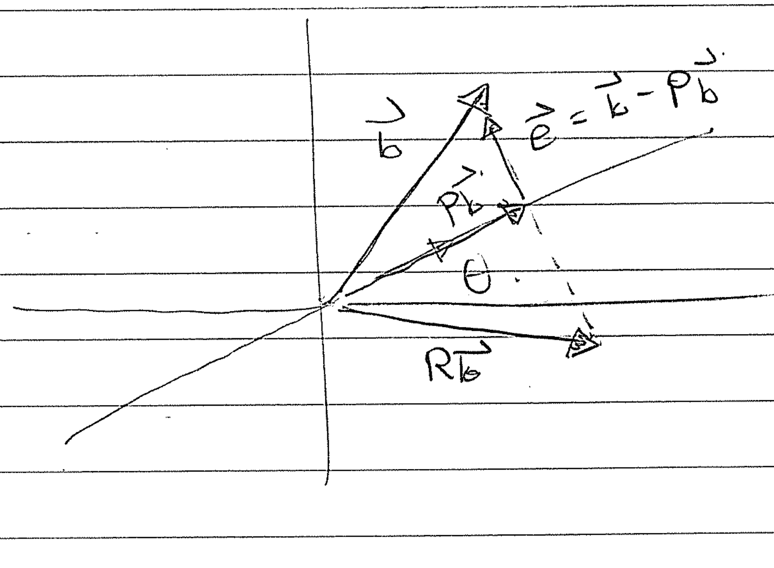
E. vectors ? Claim:



You can easily check this.

Problem A.2.

Q: What is the matrix of the reflection across the line through $(\cos\theta, \sin\theta)$?



Note that

$$R\vec{b} = \vec{b} - 2\vec{e} = \vec{b} - 2(\vec{b} - P\vec{b})$$
$$= 2P\vec{b} - \vec{b}$$
$$= 2P\vec{b} - I\vec{b}$$
$$= (2P - I)\,\vec{b}$$

$$\Longrightarrow \quad R = 2P - I.$$

$$= 2\begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2c^2 - 1 & 2cs \\ 2cs & 2s^2 - 1 \end{pmatrix}$$

E. values of R? Easy.

Suppose $P\vec{x} = \lambda\vec{x}$. Then

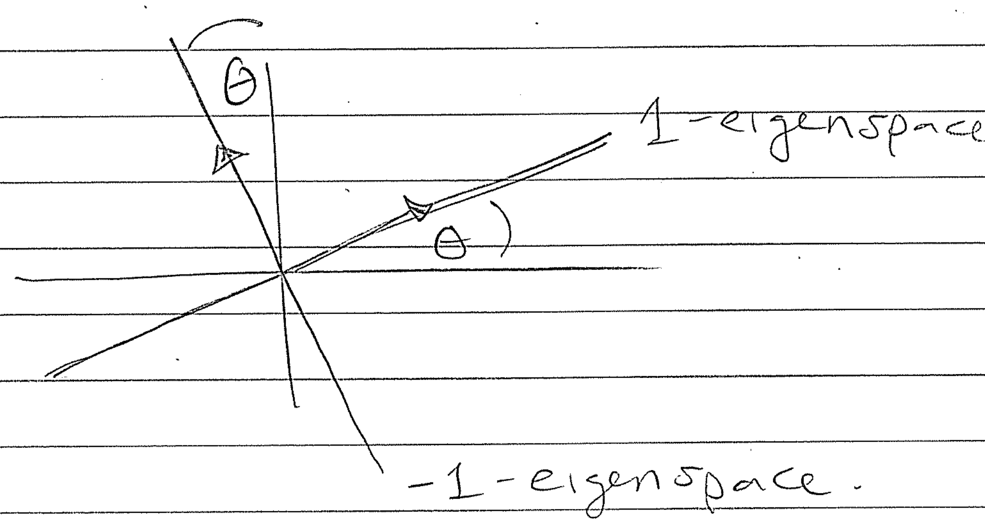
$$R\vec{x} = (2P - I)\,\vec{x}$$
$$= 2P\vec{x} - I\vec{x}$$
$$= 2\lambda\vec{x} - \vec{x}$$
$$= (2\lambda - 1)\,\vec{x}.$$

$\Rightarrow$ E.vectors of R the same as for P, but the E.values have changed from λ to $2\lambda - 1$.

P has evalues 1 and 0.

R has evalues $2(1)-1$ and $2(0)-1$
1 and -1.

Picture:



Problem 6.1.14. Find the E.values of the rotation matrix

$$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Characteristic Equation:

$$(c-\lambda)(c-\lambda) - s(-s) = 0.$$

$$c^2 - 2c\lambda + \lambda^2 + s^2 = 0.$$

$$\lambda^2 - 2c\lambda + (c^2+s^2) = 0$$

$$\lambda - 2c\lambda + 1 = 0.$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \frac{2\cos\theta \pm \sqrt{4(\cos^2\theta - 1)}}{2}.$$

$$= \frac{2\cos\theta \pm \sqrt{4(-\sin^2\theta)}}{2}.$$

$$= \frac{2\cos\theta \pm 2\sqrt{-1}\sin\theta}{2}.$$

$$= \cos\theta \pm \sqrt{-1}\sin\theta.$$

NO REAL Eigenvalues unless $\sin\theta = 0$

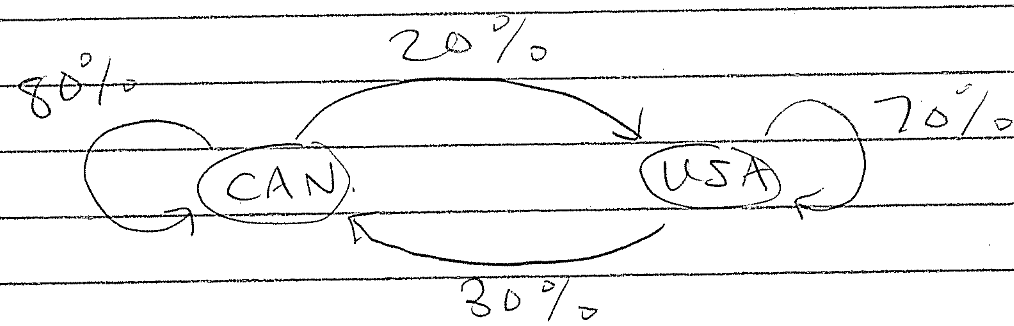
This is the meaning of complex eigenvalues:

If 2x2 matrix A has complex eigenvalues, then it has a tendency to rotate.

If A is the transition matrix of a dynamical system, then the system will oscillate.

Today: Phase Portraits

Recall the birds



and their transition matrix

$$A = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}$$

If we let c_n = # birds in CAN at year n

u_n = # birds in USA at year n.

Then we have.

$$\begin{pmatrix} c_n \\ u_n \end{pmatrix} = A \begin{pmatrix} c_{n-1} \\ u_{n-1} \end{pmatrix}$$

$$= A\,A \begin{pmatrix} c_{n-2} \\ u_{n-2} \end{pmatrix}$$

$$\vdots$$

$$= \underbrace{A\,A \cdots A}_{n \text{ times}} \begin{pmatrix} c_0 \\ u_0 \end{pmatrix}$$

$$= A^n \begin{pmatrix} c_0 \\ u_0 \end{pmatrix}$$

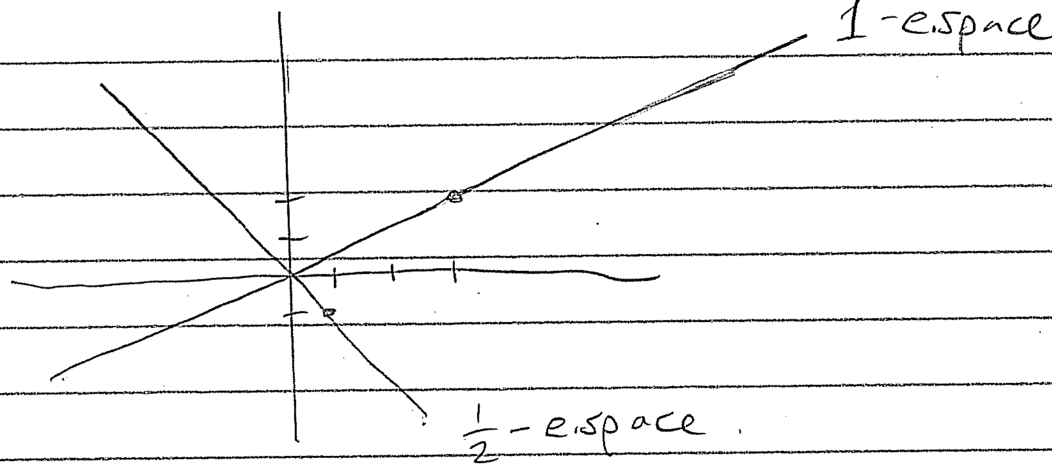
To solve this system, i.e., to find formulas for (c_n, u_n) in terms of (c_0, u_0), we must compute the eigenvalues/eigenvectors.

The e.values are 1 and $.5$.

The e.vectors are.

$$A t \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 1 \cdot t \begin{pmatrix} 3 \\ 2 \end{pmatrix} \ \& \ A t \begin{pmatrix} 1 \\ -1 \end{pmatrix} = .5 \cdot t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Picture:



This picture tells us a __lot__.
Suppose we start with $(c_0, u_0) = (10, 0)$.
This can be written in terms of eigenvectors as

$$\begin{pmatrix} 10 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 2 \end{pmatrix} + 4\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

Then we have

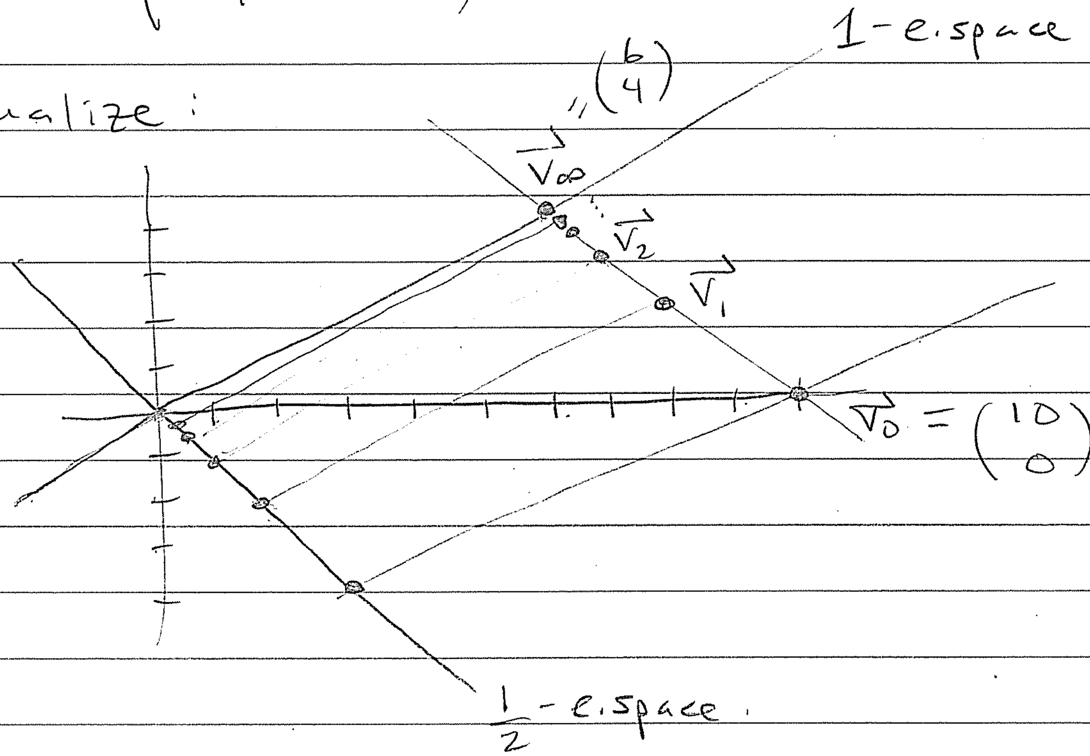
$$\begin{pmatrix} c_n \\ u_n \end{pmatrix} = A^n \begin{pmatrix} 10 \\ 0 \end{pmatrix} = A^n \left[\begin{pmatrix} 6 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ -4 \end{pmatrix} \right]$$

$$= A^n \begin{pmatrix} 6 \\ 4 \end{pmatrix} + A^n \begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

$$= 1^n \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \left(\frac{1}{2}\right)^n \begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

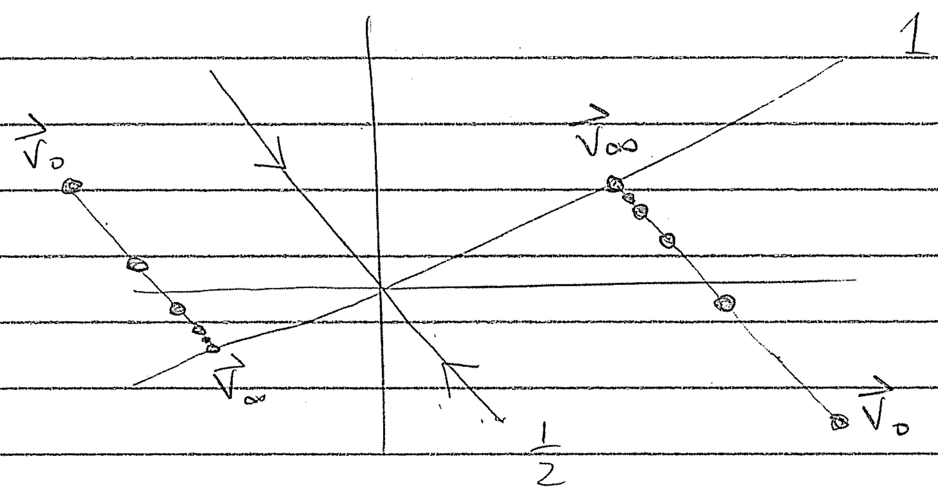
$$= \begin{pmatrix} 6 + 4/2^n \\ 4 - 4/2^n \end{pmatrix}$$

Visualize:



At each step, the state halves in the $(1,-1)$ direction and stays the same in the $(3,2)$ direction.

A general trajectory.



So the matrix A^∞ is a projection onto the line $t(3,2)$, but at a strange angle (i.e. not $90°$).

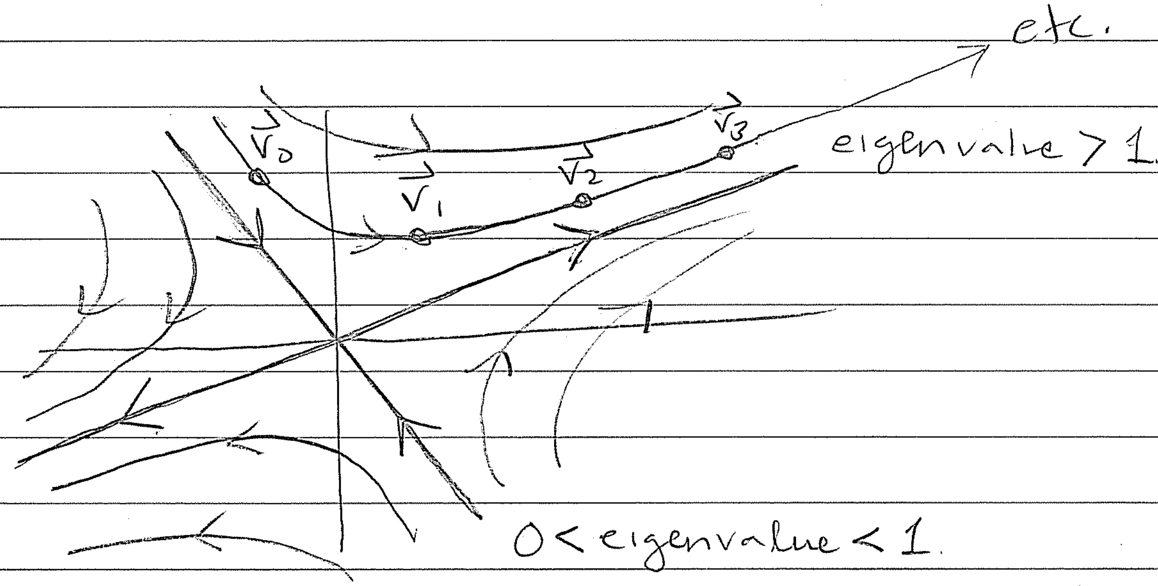
In fact, $A^\infty = \begin{pmatrix} .6 & .6 \\ .4 & .4 \end{pmatrix}$

The orthogonal projection would be

$$P = \frac{\begin{pmatrix} 3 \\ 2 \end{pmatrix}(3\ 2)}{(3\ 2)\begin{pmatrix} 3 \\ 2 \end{pmatrix}} = \frac{1}{13}\begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix} \neq A^\infty.$$

Note: A^∞ has the same e.vectors but with e.values $1^\infty = \underline{1}$ and $(.5)^\infty = 0$

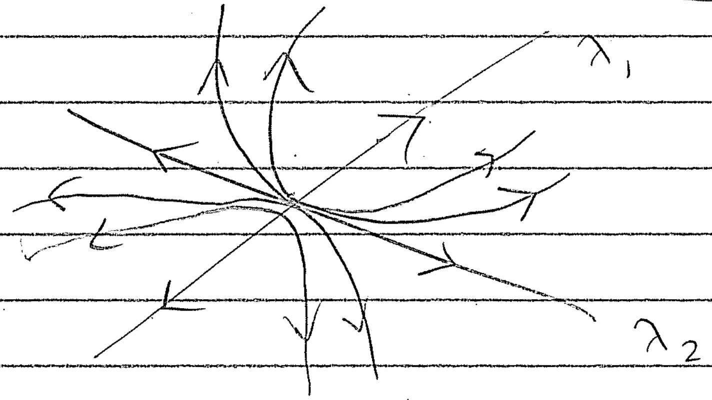
Q: What if we had a matrix with <u>this</u> eigen-information:



This is called a "phase portrait"
It shows us the <u>typical</u> <u>trajectories</u>.

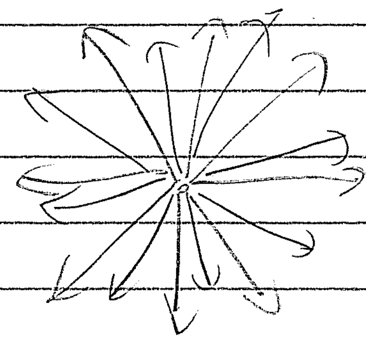
The evalues/evectors determine the behavior of the system.

Other Possibilities: $\boxed{\lambda_1 > \lambda_2 > 1}$

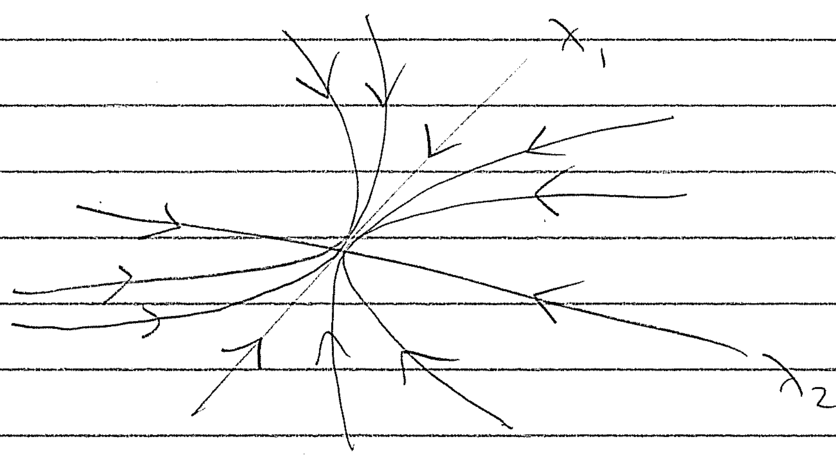


$\boxed{\lambda_1 = \lambda_2 > 1}$

Expands evenly in all directions



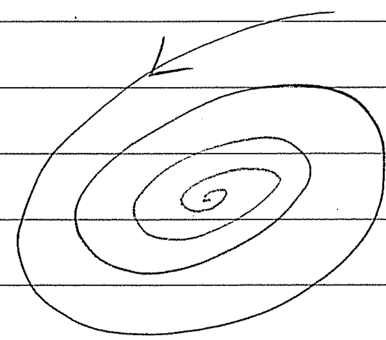
$\boxed{0 < \lambda_1 < \lambda_2 < 1}$



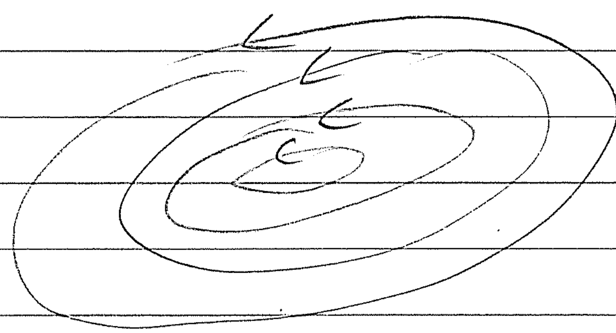
What if λ_1, λ_2 are complex?

Then the system will oscillate

$|\lambda_1| = |\lambda_2| < 1$

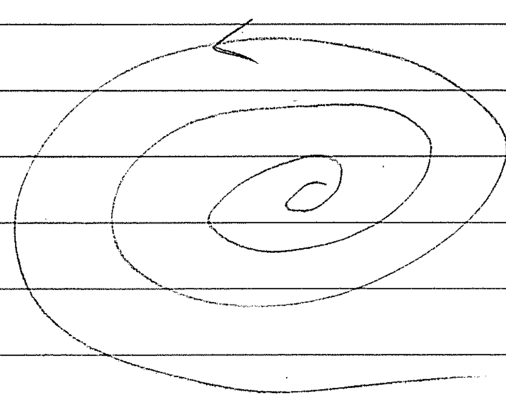


$|\lambda_1| = |\lambda_2| = 1$



closed orbits

$|\lambda_1| = |\lambda_2| > 1$



Today: The phase portrait of Fibonacci numbers.

Recall the Fibonacci numbers

$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

They are defined by initial conditions

$$\begin{pmatrix} F_1 \\ F_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and second-order recurrence

$$F_{n+2} = F_{n+1} + F_n \quad \text{for all } n \geq 0.$$

which we can write as a 2×2 matrix equation (dynamical system)

$$\begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$$

If we let $\vec{V}_n = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$ then we can translate this as:

Initial condition $\vec{V}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Recurrence $\vec{V}_{n+1} = A \vec{V}_n$ for all $n \geq 0$,

where $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

<u>Problem</u>: <u>Solve</u> the system.

Compute eigenvalues.

$$(1-\lambda)(0-\lambda) - 1 \cdot 1 = 0$$
$$-\lambda + \lambda^2 - 1 = 0$$
$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{(-1)^2 - 4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Call these $\alpha = \frac{1+\sqrt{5}}{2}$

$\beta = \frac{1-\sqrt{5}}{2}$

and observe that $\alpha^2 - \alpha - 1 = 0$

$$\beta^2 - \beta - 1 = 0$$
$$\alpha + \beta = 1$$
$$\alpha\beta = -1$$

Compute eigenvectors:

$\lambda = \alpha$: $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \alpha\begin{pmatrix} x \\ y \end{pmatrix}$.

$$\begin{pmatrix} 1-\alpha & 1 \\ 1 & -\alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\left(\begin{array}{cc|c} 1-\alpha & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} \beta & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & 1/\beta & 0 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -\alpha & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Solution $x - \alpha y = 0$

$$x = \alpha y.$$

This is the line $\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} \alpha \\ 1 \end{pmatrix}$.

$\lambda = \beta$: $$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \beta \begin{pmatrix} x \\ y \end{pmatrix}.$$

$$\begin{pmatrix} 1-\beta & 1 \\ 1 & -\beta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1-\beta & 1 & 0 \\ 1 & -\beta & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1-\beta & 1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

$$\longrightarrow \left(\begin{array}{cc|c} \alpha & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1/\alpha & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & -\beta & 0 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow \begin{array}{l} x - \beta y = 0 \\ x = \beta y. \end{array}$$

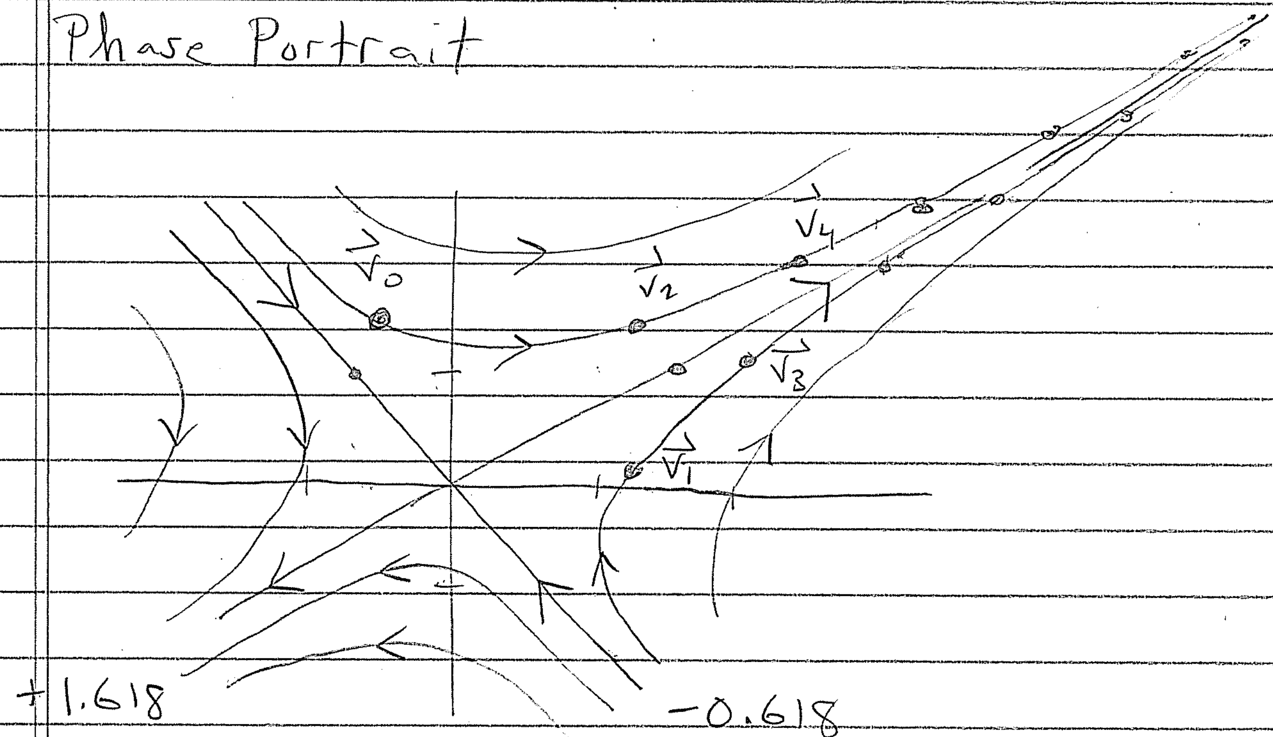
This is the line $\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} \beta \\ 1 \end{pmatrix}$.

Now observe

$$\alpha = \frac{1+\sqrt{5}}{2} \approx 1.618 \qquad \text{"the golden ratio"}$$

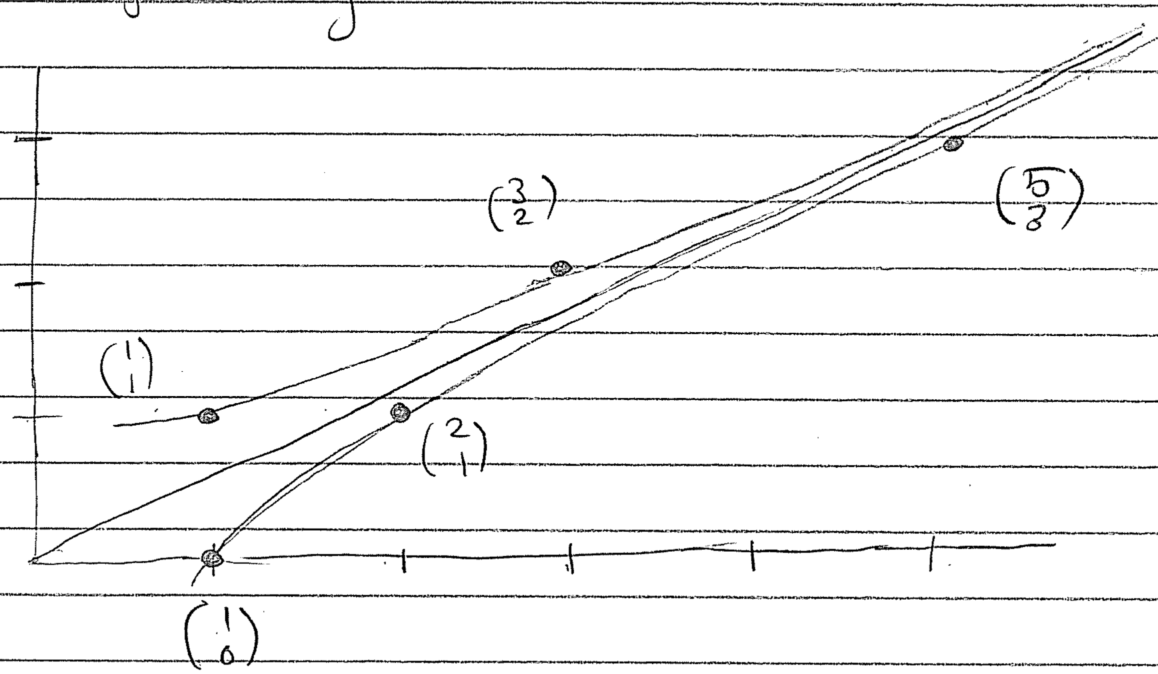
$$\beta = \frac{1-\sqrt{5}}{2} \approx -0.618$$

Phase Portrait



But the -0.618 hops back and forth.

Our Trajectory.



The points $\vec{v}_n = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$ get very close to the line $t\begin{pmatrix} \alpha \\ 1 \end{pmatrix}$. In other words,

$$\frac{F_{n+1}}{F_n} \approx \frac{\alpha}{1} \approx 1.618 \quad \text{"golden ratio"}$$

This means that

$$F_n \approx C\,\alpha^n = C\,(1.618)^n$$

for some constant C.
What is the constant?

Step 1: Express $\vec{v}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in terms of eigenvectors:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} \alpha \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}}\begin{pmatrix} \beta \\ 1 \end{pmatrix}$$

Details omitted.

Step 2: Apply A^n.

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$= A^n \left[\frac{1}{\sqrt{5}} \begin{pmatrix} \alpha \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}} \begin{pmatrix} \beta \\ 1 \end{pmatrix} \right]$$

$$= \frac{1}{\sqrt{5}} A^n \begin{pmatrix} \alpha \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}} A^n \begin{pmatrix} \beta \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \alpha^n \begin{pmatrix} \alpha \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}} \beta^n \begin{pmatrix} \beta \\ 1 \end{pmatrix}$$

$$\Longrightarrow F_n = \frac{1}{\sqrt{5}} \alpha^n - \frac{1}{\sqrt{5}} \beta^n.$$

$$= \frac{1}{\sqrt{5}} (1.618)^n - \frac{1}{\sqrt{5}} (-0.618)^n$$

$$\approx \frac{1}{\sqrt{5}} (1.618)^n$$

Because $\frac{1}{\sqrt{5}}(-0.618)^n \longrightarrow 0$

as $n \longrightarrow \infty$.