June 5 - June 9

Last time I stated the "Central Problem of Linear Algebra":

☆ To solve a system of m simultaneous linear equations in n unknowns.

We will write the general system as

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

where $x_1, \ldots, x_n$ are variables and $a_{11}, \ldots, a_{mn}$ & $b_1, \ldots, b_m$ are constants.

There are two different ways to visualize a linear system. Gilbert Strang calls them the "row picture" and the "column picture".

(1) The Row Picture.

The m simultaneous linear equations in n variables represent the intersection of m hyperplanes in n-dimensional space.

Intuition: If the equations are "random" or "generic" then the solution will be an

$(n-m)$-dimensional plane.

If $m > n$ then there is "probably"

NO SOLUTION.

Example: 3 planes in 3D probably meet at a point. 4 planes in 3D probably don't meet anywhere [the first 3 probably meet at a point and then the 4th probably doesn't contain this point].

(2) The Column Picture.

We can rewrite the original system of m linear equations as one vector equation:

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\boxed{x_1 \vec{a}_{*1} + x_2 \vec{a}_{*2} + \cdots + x_n \vec{a}_{*n} = \vec{b}}$$

Problem: Combine n given vectors in m-dimensional space to reach a given target vector. In other words: starting at $\vec{0}$, how far do you have to travel in the directions $\vec{a}_{*1}, \vec{a}_{*2}, \ldots, \vec{a}_{*n}$ in order to reach the restaurant at $\vec{b}$?

Since this problem is mathematically equivalent to (1) we can transfer some of our intuition.

Example: If I give you 3 direction vectors $\vec{a}_{*1}, \vec{a}_{*2}, \vec{a}_{*3}$ and a target vector $\vec{b}$ in 4D space, can you find numbers x_1, x_2, x_3 such that

$$x_1 \vec{a}_{*1} + x_2 \vec{a}_{*2} + x_3 \vec{a}_{*3} = \vec{b} \; ?$$

Probably not! There are two reasons:

1. The vectors $x_1 \vec{a}_{*1} + x_2 \vec{a}_{*2} + x_3 \vec{a}_{*3}$ probably form a 3-plane in 4D and the point $\vec{b}$ is probably <u>not in this 3-plane</u>.

2. The row picture of this problem is the intersection of 4 planes in 3D and we already agreed that this probably has <u>no solution</u>.

The pictures ① & ② are quite valuable for interpreting solutions or guessing the form of the solution. But they don't help us to actually <u>compute</u> the solution. For that we need a specific algebraic technique; and we are lucky to have one.

Our technique is called "Gaussion Elimination" and we've already seen it in action. I'll be a bit more explicit today.

Example of Gaussian Elimination:

$$\begin{cases} x_1 + 3x_2 + 0 + 2x_4 = 1 & (1) \\ 0 + 0 + x_3 + 4x_4 = 6 & (2) \\ x_1 + 3x_2 + x_3 + 6x_4 = 7 & (3) \end{cases}$$

Use the "pivot" x_1 in (1) to eliminate x_1 from (2) & (3). Luckily, equation (2) already has no x_1.

$$\begin{cases} x_1 + 3x_2 + 0 + 2x_4 = 1 & (1)' = (1) \\ 0 + 0 + x_3 + 4x_3 = 6 & (2)' = (2) \\ 0 + 0 + x_3 + 4x_3 = 6 & (3)' = (3) - 1(1) \end{cases}$$

Now we look for a pivot in the x_2 column but there isn't one! So we move on the x_3 column. We will use the pivot x_3 in (2)' to eliminate the x_3 from (3)'.

$$\begin{cases} x_1 + 3x_2 + 0 + 2x_4 = 1 & \quad (1)'' = (1)' \\ 0 + 0 + x_3 + 4x_4 = 6 & \quad (2)'' = (2)' \\ 0 + 0 + 0 + 0 = 0 & \quad (3)'' = (3)' - 1(2)' \end{cases}$$

Now we look for a pivot in the x_4 column but there isn't one! Oh well. Now our system is in "row echelon form" (REF).

[Note : echelon = staircase].

The final step is to multiply equations by numbers so that the pivot terms have coefficient 1. Then we perform "backwards elimination" to eliminate the terms <u>above</u> our pivots. Since both of these steps are already done (luckily) we can say that our system is in "reduced row echelon form" (RREF).

Once the system is in RREF it becomes easy to read off the solution. In our case we have

pivot variables : x_1, x_3
free variables : x_2, x_4

The solution is

$$x_1 = 1 - 3x_2 - 2x_4$$
$$x_2 = x_2$$
$$x_3 = 6 - 4x_4$$
$$x_4 = x_4$$

which can be written in vector form as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 - 3x_2 - 2x_4 \\ 0 + 1x_2 + 0x_4 \\ 6 + 0x_2 - 4x_4 \\ 0 + 0x_2 + 1x_4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 6 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ 4 \\ 1 \end{pmatrix}.$$

Note that the solution is "2-dimensional" because it has two free variables.

Recall from last time: We considered the linear system

$$\begin{cases} x_1 + 3x_2 + 0 + 2x_4 = 1 & (1) \\ 0 + 0 + x_3 + 4x_4 = 6 & (2) \\ x_1 + 3x_2 + x_3 + 6x_4 = 7 & (3) \end{cases}$$

We performed "Gaussian elimination" to put the system in the form

$$\begin{cases} x_1 + 3x_2 + 0 + 2x_4 = 1 \\ 0 + 0 + x_3 + 4x_4 = 6 \\ 0 + 0 + 0 + 0 = 0 \end{cases}$$

We called this the "reduced row echelon form" (RREF) of the system.

The variables in the corners of the staircase (i.e. x_1 & x_3) are called pivot variables and all other variables (i.e. x_2 & x_4) are called free variables.

Finally we can write down the solution in terms of the free variables:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 - 3x_2 - 2x_4 \\ x_2 \\ 6 - 4x_4 \\ x_4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 3x_2 - 2x_4 \\ 0 + 1x_2 + 0x_4 \\ 6 + 0x_2 - 4x_4 \\ 0 + 0x_2 + 1x_4 \end{pmatrix}$$

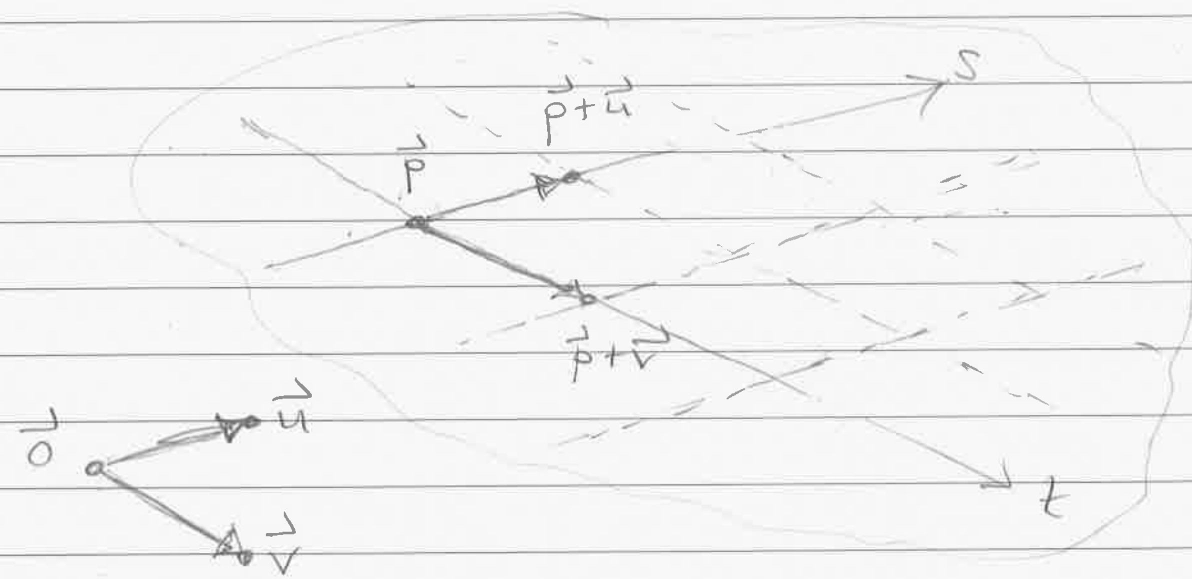
$$= \begin{pmatrix} 1 \\ 0 \\ 6 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix}.$$

To clean this up let's define $\vec{p} = (1,0,6,0)$, $\vec{u} = (-3,1,0,0)$, $\vec{v} = (-2,0,-4,1)$, $x_2 = s$ & $x_4 = t$.

Then the solution is

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \vec{p} + s\vec{u} + t\vec{v}}$$

Picture: This is the 2-dimensional plane in 4D containing the point $\vec{p}$ and spanned by the vectors $\vec{u}$ & $\vec{v}$.



This plane does not contain the origin $\vec{0}$.

We can think of the 2-plane $\vec{p}+s\vec{u}+t\vec{v}$ as the interesection of the three hyperplanes defined by equations (1), (2) & (3).

This is the row picture

Q: We expect three hyperplanes in 4D to intersect in a line (1-plane). What went wrong?

A: While performing elimination we found the relationship

$$(1) + (2) = (3).$$

This means that any solution to the first two equations is also a solution to the third. Geometrically, the intersection hyperplanes (1) & (2) is accidentally contained in the hyperplane (3).

That means there must also be something wrong with the column picture. Let's see what it is. The system (*) becomes one vector equation:

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 7 \end{pmatrix}.$$

AHA, I see the problem! The vectors

$$\begin{pmatrix}1\\0\\1\end{pmatrix} \;\&\; \begin{pmatrix}3\\0\\3\end{pmatrix}$$

are in the same direction, so one of them was completely unnecessary.

That was a human introduction to Gaussian elimination. Now let me show you how I would tell it to a computer.

Given a system of equations, there are three operations we can do that will yield an equivalent system (i.e. a system with the same solution):

(A) Swap two equations

(B) Replace equation (i) by c (i) where c is a nonzero constant.

(C) Replace equation (i) by (i) − c (j) where c is any constant and (j) is any other equation.

We call (A), (B), (C) the elementary row operations (EROs). The goal of Gaussian elimination is to perform a sequence of EROs to put a linear system in a nice, standard form (the RREF).

[Most computers have a button to do this.]

Here's (one version of) the algorithm:

- Do (A) to get a nonzero pivot in the top left corner. If this is impossible, move one column to the right. If that's impossible, STOP.

- Do (B) to turn the pivot into a 1.

- Do (C) to eliminate all entries below the pivot.

- Repeat the process on the subsystem below and to the right of the pivot.

Now the system looks something like this:

$$\left\{\begin{array}{cccccccccc} 0 & 0 & 1 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right.$$

- Finally, do (C) to eliminate all entries <u>above</u> the pivots, working from the bottom right to the top left.

NOW the system is in RREF, which looks something like this:

$$\left\{\begin{array}{cccccccccc} 0 & 0 & 1 & * & 0 & * & * & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right.$$

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Performing Gaussian elimination is a job best left to computers, but we <u>will</u> practice doing it by hand on some small systems. [see HW 3]

Last time I finally defined the method of "Gaussian elimination" in all its gory details. This method was invented by Carl Friedrich Gauss around 1800 in order to compute the orbits of various celestial bodies. However, a similar method already appeared in China in the "Nine Chapters on the Mathematical Art" (263 A.D.).

The algorithm is best suited for computers but we can still compute some small systems by hand. Today we'll get some practice with this.

Example 1: Solve the system.

$$\begin{cases} 0 + y + z = 1 & (1) \\ x + y + z = 2 & (2) \\ 2x + 0 + 3z = 1 & (3) \end{cases}$$

First swap (1) & (2) to get a pivot in the top left.

$$\begin{cases} x + y + z = 2 & ①' = ② \\ 0 + y + z = 1 & ②' = ① \\ 2x + 0 + 3z = 1 & ③' = ③. \end{cases}$$

The pivot already has coefficient 1. ✓
Now eliminate below the x pivot.

$$\begin{cases} x + y + z = 2 & ①'' = ①' \\ 0 + y + z = 1 & ②'' = ②' \\ 0 - 2y + z = -3 & ③'' = ③' - 2①' \end{cases}$$

Now recursively apply the method to the subsystem $②''$ & $③''$ that involves only y & z.

$$\begin{cases} x + y + z = 2 & ①''' = ①'' \\ 0 + y + z = 1 & ②''' = ②'' \\ 0 + 0 + 3z = -1 & ③''' = ③'' + 2②'' \end{cases}$$

Divide equation $③'''$ by 3 to get the pivot 1. Now the system is in "row echelon form" (REF).

$$\begin{cases} x + y + z = 2 & ①'''' = ①''' \\ 0 + y + z = 1 & ②'''' = ②''' \\ 0 + 0 + z = -1/3 & ③'''' = 1/3\, ③'''. \end{cases}$$

To put the system in reduced row echelon form (RREF) we first eliminate above the z pivot.

$$\begin{cases} x + y + 0 = 7/3 & 1''''' = 1'''' - 3''''' \\ 0 + y + 0 = 4/3 & 2''''' = 1'''' - 2'''' \\ 0 + 0 + z = -1/3 & 3''''' = 3'''' \end{cases}$$

Finally, we eliminate above the y pivot.

$$\begin{cases} x + 0 + 0 = 1 & 1'''''' = 1''''' - 2''''' \\ 0 + y + 0 = 4/3 & 2'''''' = 2''''' \\ 0 + 0 + z = -1/3 & 3'''''' = 2''''' \end{cases}$$

This is the RREF, and now the solution is obvious:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4/3 \\ -1/3 \end{pmatrix}.$$

Row Picture: The three planes 1, 2, 3 meet at the single point $(1, 4/3, -1/3)$.

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Column Picture: We can reach the point (1, 2, 1) by combining the three columns as follows.

$$1\begin{pmatrix}0\\1\\2\end{pmatrix}+\frac{4}{3}\begin{pmatrix}1\\1\\0\end{pmatrix}-\frac{1}{3}\begin{pmatrix}1\\1\\3\end{pmatrix}=\begin{pmatrix}1\\2\\1\end{pmatrix}.$$

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As we see, the notation gets quite cumbersome. So in the next example let's streamline the notation by throwing away all unnecessary symbols.

Example 2: Solve the system.

$$\begin{cases} 2x_1 - 3x_2 - x_3 + 2x_4 + 3x_5 = 4 & (1) \\ 4x_1 - 4x_2 - x_3 + 4x_4 + 11x_5 = 4 & (2) \\ 2x_1 - 5x_2 - 2x_3 + 2x_4 - x_5 = 9 & (3) \\ 0 + 2x_2 + x_3 + 0 + 4x_5 = 5 & (4) \end{cases}$$

Instead we'll write it like this:

$$\left(\begin{array}{ccccc|c} 2 & -3 & -1 & 2 & 3 & 4 \\ 4 & -4 & -1 & 4 & 11 & 4 \\ 2 & -5 & -2 & 2 & -1 & 9 \\ 0 & 2 & 1 & 0 & 4 & 5 \end{array}\right) \begin{array}{c} (1) \\ (2) \\ (3) \\ (4). \end{array}$$

This is called the "augmented matrix" notation. Now we perform Gaussian elimination as usuall. [Actually, I'll avoid scaling the pivots to 1 until the end because I don't like fractions.]

$$\left(\begin{array}{ccccc|c} 2 & -3 & -1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 0 & 5 & -4 \\ 0 & -2 & -1 & 0 & -4 & 5 \\ 0 & 2 & 1 & 0 & 4 & 5 \end{array}\right) \quad \begin{array}{l} ① \to ① \\ ② \to ② - ① \\ ③ \to ③ - ① \\ ④ \to ④ \end{array}$$

$$\left(\begin{array}{ccccc|c} 2 & -3 & -1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 0 & 5 & -4 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{array}\right) \quad \begin{array}{l} ① \to ① \\ ② \to ② \\ ③ \to ③ + ② \\ ④ \to ④ - ② \end{array}$$

$$\left(\begin{array}{ccccc|c} 2 & -3 & -1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 0 & 5 & -4 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} ① \to ① \\ ② \to ② \\ ③ \to ③ \\ ④ \to ④ + ③ \end{array}$$

REF
$$\left(\begin{array}{ccccc|c} 1 & -3/2 & -1/2 & 1 & 3/2 & 2 \\ 0 & 1 & 1/2 & 0 & 5/2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} ① \to \tfrac{1}{2}① \\ ② \to \tfrac{1}{2}② \\ ③ \to ③ \\ ④ \to ④ \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & -3/2 & -1/2 & 1 & 0 & 1/2 \\ 0 & 1 & 1/2 & 0 & 0 & -9/2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} (1) \to (1) - \frac{3}{2}(3) \\ (2) \to (2) - \frac{5}{2}(3) \\ (3) \to (3) \\ (4) \to (4) \end{array}$$

RREF
$$\left(\begin{array}{ccccc|c} 1 & 0 & 1/4 & 1 & 0 & -25/4 \\ 0 & 1 & 1/2 & 0 & 0 & -9/2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} (1) \to (1) + \frac{3}{2}(2) \\ (2) \to (2) \\ (3) \to (3) \\ (4) \to (4) \end{array}$$

Translating back to old notation, the RREF of the system is

$$\begin{cases} x_1 + 0 + \frac{1}{4}x_3 + x_4 + 0 = -25/4 \\ 0 + x_2 + \frac{1}{2}x_3 + 0 + 0 = -9/2 \\ 0 + 0 + 0 + 0 + x_5 = 1 \\ 0 + 0 + 0 + 0 + 0 = 0 \end{cases}$$

Pivot variables : x_1, x_2, x_5
Free variables : x_3, x_4.

So let's define $s := x_3$, $t := x_4$ and then express the solution in terms of the parameters s & t.

The solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -25/4 - 1/4\, x_3 - x_4 \\ -9/2 - 1/2\, x_3 \\ x_3 \\ x_4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -25/4 - 1/4\, s - 1t \\ -9/2 - 1/2\, s + 0t \\ 0 + 1s + 0t \\ 0 + 0s + 1t \\ 1 + 0s + 0t \end{pmatrix}$$

$$\begin{pmatrix} -25/4 \\ -9/2 \\ 0 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} -1/4 \\ -1/2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

This is a parametrized 2D plane living in 5D space. We expected a line $(5-4=1)$ but we got a plane, so there must be some relationship among the equations.

Last time we looked at the following system of 4 linear equations in 5 unknowns:

$$\begin{cases} 2x_1 - 3x_2 - x_3 + 2x_4 + 3x_5 = 4 & (1) \\ 4x_1 - 4x_2 - x_3 + 4x_4 + 11x_5 = 4 & (2) \\ 2x_1 - 5x_2 - 2x_3 + 2x_4 - x_5 = 9 & (3) \\ 0 + 2x_2 + x_3 + 0 + 4x_5 = 5 & (4) \end{cases}$$

We dropped all unnessary symbols to write this as an "augmented matrix":

$$\left(\begin{array}{ccccc|c} 2 & -3 & -1 & 2 & 3 & 4 \\ 4 & -4 & -1 & 4 & 11 & 4 \\ 2 & -5 & -2 & 2 & -1 & 9 \\ 0 & 2 & 1 & 0 & 4 & 5 \end{array}\right)$$

Then we performed Gaussian elimination to put the matrix in RREF:

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1/4 & 1 & 0 & -25/4 \\ 0 & 1 & 1/2 & 0 & 0 & -9/2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Finally, we turned this back into a system of equations

$$\begin{cases} x_1 + \frac{1}{4}x_3 + x_4 = -25/4 \\ x_2 + \frac{1}{2}x_3 = -9/2 \\ x_5 = 1 \\ 0 = 0 \end{cases}$$

and then read off the solution.
Naming the free variables $x_3 = s$ & $x_4 = t$ gives us

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -\frac{25}{4} - \frac{1}{4}s - t \\ -9/2 - 1/2\, s \\ s \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} -25/4 \\ -9/2 \\ 0 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} -1/4 \\ -1/2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

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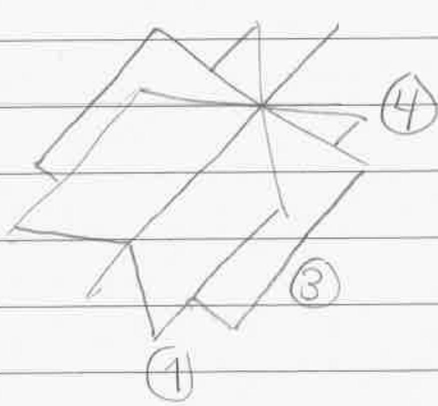
Now let's interpret the solution.

Row Picture: The intersection of the 4 hyperplanes 1, 2, 3, 4 is a 2-dimensional plane living in 5D space.

This is not what we expected. [With $m=4$ equations in $n=5$ unknowns we expect a $(n-m)=(5-4)=1$ dimensional solution.] So there must have been some relationship among the equations. Sure enough, we have

$$1 = 3 + 4,$$

which means that any one of these three equations can be thrown away without changing the solution. Geometrically, the intersection of any two of these hyperplanes is contained in the third. My mental picture looks like this



even though I know this picture has the wrong dimension. [The correct picture is three 4-planes meeting at a 3-plane in 5D space, which I can't draw.]

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Column Picture: We are trying to hit the point $\vec{b} = (4, 4, 9, -5)$ in 5D space by <u>combining</u> the five vectors

$$\vec{a}_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \\ 0 \end{pmatrix}, \vec{a}_2 = \begin{pmatrix} -3 \\ -4 \\ -5 \\ 2 \end{pmatrix}, \vec{a}_3 = \begin{pmatrix} -1 \\ -1 \\ -2 \\ 1 \end{pmatrix}, \vec{a}_4 = \begin{pmatrix} 2 \\ 4 \\ 2 \\ 0 \end{pmatrix}, \vec{a}_5 = \begin{pmatrix} 3 \\ 11 \\ -1 \\ 4 \end{pmatrix}.$$

We know that there must be some relationship among these vectors [because the solution doesn't have the expected dimension] and indeed there is:

$$\vec{a}_1 = \vec{a}_4 .$$

This means that <u>if</u> the problem has a solution, then it must have <u>infinitely many</u> solutions.

Indeed, suppose that

$$(*).\quad x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 + x_4\vec{a}_4 + x_5\vec{a}_5 = \vec{b}$$

is a solution. Then I claim that for any number k we have another solution

$$(x_1+k,\ x_2,\ x_3,\ x_4-k,\ x_5).$$

Proof: Assuming $(*)$ is true we have

$$(x_1+k)\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 + (x_4-k)\vec{a}_4 + x_5\vec{a}_5$$

$$= (x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 + x_4\vec{a}_4 + x_5\vec{a}_5) + (k\vec{a}_1 - k\vec{a}_4)$$

$$= \vec{b} + \vec{0} = \vec{b},$$

because $\vec{a}_1 = \vec{a}_4$ and hence

$$k\vec{a}_1 - k\vec{a}_4 = k\vec{a}_1 - k\vec{a}_1 = \vec{0}.$$

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That's enough interpretation for today.

Let's summarize what we know about linear systems and Gaussian elimination.

1. A linear system looks like this:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

2. We can express it as an augmented matrix by dropping all the unnecessary symbols:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

3. After performing Gaussian elimination we obtain the reduced row echelon form (RREF) which looks like this:

$$\left(\begin{array}{cccccccc|c} 0 & 1 & * & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{array}\right)$$

4. If we obtain the equation $0 = *$ where $*$ is not zero, then the system has NO SOLUTION.

5. Otherwise, we let $t_1, t_2, \ldots, t_f$ be the free variables and we express the solution as

$$\vec{x} = \vec{p} + t_1 \vec{u}_1 + t_2 \vec{u}_2 + \cdots + t_f \vec{u}_f$$

for some point $\vec{p}$ and some vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_f$. The solution is on f-dimensional plane living in n-dimensional space.

For the example in part 3, we have pivot variables

$$x_2, x_4, x_7$$

and free variables

$$x_1, x_3, x_5, x_6, x_8.$$

The solution is a 5-dimensional plane living in 8-dimensional space.

Now you have seen everything there is to see about Gaussian elimination. We'll let it sink in for a little while and then we'll move on to something else.

Today: HW3 Discussion.

Problem 1': Why do I say that a hyperplane is "flat"?

Let $\vec{a}$ be a vector in n-dimensional space and let b be a constant. Then the equation

$$\boxed{\vec{a} \cdot \vec{x} = b}$$

defines the <u>hyperplane</u> perpendicular to $\vec{a}$ that has minimum distance $b/\|\vec{a}\|$ from the origin.

Suppose that $\vec{x}_1$ & $\vec{x}_2$ are two points on this hyperplane. That is, suppose that the equations

$$\vec{a} \cdot \vec{x}_1 = b \quad \& \quad \vec{a} \cdot \vec{x}_2 = b$$

are both true.

Then I claim that the midpoint

$$\frac{1}{2}(\vec{x}_1 + \vec{x}_2)$$

is also on the hyperplane.

Proof: We have

$$\vec{a} \cdot \left(\frac{1}{2}(\vec{x}_1 + \vec{x}_2) \right)$$

$$= \frac{1}{2} \vec{a} \cdot (\vec{x}_1 + \vec{x}_2)$$

$$= \frac{1}{2} \left(\vec{a} \cdot \vec{x}_1 + \vec{a} \cdot \vec{x}_2 \right)$$

$$= \frac{1}{2}(b + b)$$

$$= \frac{1}{2}(2b) = b \checkmark$$

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Note that this same fact is <u>not</u> true for curvy shapes like a paraboloid or the surface of a sphere. [The midpoint of two points on the surface of a sphere will be <u>inside</u> the sphere, <u>not</u> on the surface.]

More generally, the set of points

$$s\vec{x}_1 + t\vec{x}_2 \quad \text{with} \quad s+t=1$$

is the unique line in n-dimensional space containing the two points $\vec{x}_1$ & $\vec{x}_2$.

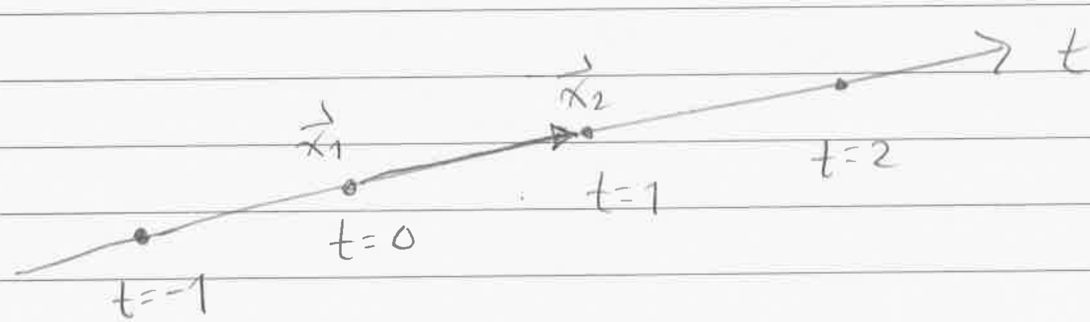
Q: If it's a line, why does it have two free parameters?

A: It doesn't! The equation $s+t=1$ means that $s=1-t$, so we can express the line as

$$s\vec{x}_1 + t\vec{x}_2 = (1-t)\vec{x}_1 + t\vec{x}_2$$

$$= \vec{x}_1 + t(\vec{x}_2 - \vec{x}_1).$$

This is the line containing the point $\vec{x}_1$ and parallel to the vector $\vec{x}_2 - \vec{x}_1$.



Now, if $\vec{x}_1$ & $\vec{x}_2$ are two points on the hyperplane $\vec{a} \cdot \vec{x} = b$, I claim that the whole line $\vec{x}_1 + t(\vec{x}_2 - \vec{x}_1)$ lives in the hyperplane.

Proof: Assume that $\vec{a} \cdot \vec{x}_1 = b$ & $\vec{a} \cdot \vec{x}_2 = b$. Then for all values of t we have

$$\vec{a} \cdot \left(\vec{x}_1 + t(\vec{x}_2 - \vec{x}_1) \right)$$

$$= \vec{a} \cdot \vec{x}_1 + t\, \vec{a} \cdot (\vec{x}_2 - \vec{x}_1)$$

$$= \vec{a} \cdot \vec{x}_1 + t\left(\vec{a} \cdot \vec{x}_1 - \vec{a} \cdot \vec{x}_2 \right)$$

$$= b + \cancel{t(b - b)} = b. \checkmark$$

This is really what I mean when I say that a hyperplane is "flat".

But even more is true. Suppose that we have a system of hyperplanes

$$\vec{a}_1 \cdot \vec{x} = b_1, \ \vec{a}_2 \cdot \vec{x} = b_2, \ \ldots, \ \vec{a}_m \cdot \vec{x} = b_m.$$

If the two points $\vec{x}_1$ & $\vec{x}_2$ lie in the intersection of the hyperplanes, then they lie in each individual hyperplane, so the line $\vec{x}_1 + t(\vec{x}_2 - \vec{x}_1)$ lies in each individual hyperplane, so the line lies in the intersection of the hyperplanes.

We conclude that any intersection of hyperplanes is also "flat".

In particular [see HW 1(b)], if 25 hyperplanes in 12-dimensional space meet at two given points $\vec{x}_1$ & $\vec{x}_2$ then they also meet at

every point of the line $\vec{x}_1 + t(\vec{x}_2 - \vec{x}_1)$!

[including, for example, the midpoint of $\vec{x}_1$ & $\vec{x}_2$ (when $t = 1/2$)]

Starting on Friday we will begin developing a languge that makes it much easier to say these things: the languge of "matrix algebra".

Problem 3'. Consider the system

$$\begin{cases} x + y + z = 2 & (1) \\ x + 2y + z = 3 & (2) \\ 2x + 3y + 2z = c & (3). \end{cases}$$

The planes (1) & (2) meet in a line L, which we can find by reducing the subsystem of equations (1) & (2):

$$\begin{cases} x + y + z = 2 & (1) \to (1) \\ 0 + y + 0 = 1 & (2) \to (2) - (1) \end{cases}$$

$$\begin{cases} x + 0 + z = 1 & (1) \to (1) - (2) \\ 0 + y + 0 = 1 & (2) \to (2) \end{cases}$$

Let $t := z$. Then the line L is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-t \\ 1 \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

The question is: how does the plane (3) meet the line L?

Well, there are three possible cases:

i) (3) contains the line L,

ii) (3) intersects L at a point,

iii) (3) never meets L (i.e. the plane (3) is parallel to the line L).

On HW3 you found that

i) happens when $c = 5$,

ii) never happens,

iii) happens when $c \neq 5$.

See the HW3 solutions for a beautiful picture of the situations i) & iii).