May 22 - May 26

MTH 210
Intro. to Linear Algebra.

Drew Armstrong
armstrong@math.miami.edu
Ungar 437.

All lecture notes and course information will be posted on my webpage:

www.math.miami.edu/~armstrong

For extra reading and practice problems I recommend Gilbert Strang's "Intro. to Linear Algebra", 4th ed., but this text is not required.

I also have lecture notes from Spring 2013 posted on my webpage.

Your grade will be based on:

Homework	1/3
Quizzes	1/3
Final Exam	1/3

What is this course about?

Linear Algebra is the common denominator of all mathematics. From the most pure to the most applied, if you use mathematics then you <u>will</u> use Linear Algebra.

The importance of Calculus has plateaued but Linear Algebra continues to gain ground as computers and data become more important.

Before discussing applications of Linear Algebra we must first develop the languge, which is based on "matrices" and "vectors". These, in turn, are based on "coordinate geometry", so that's where we'll begin.

BEGIN.

The subject of geometry is fundamentally about points in space. But what is "space", and what is a "point"?

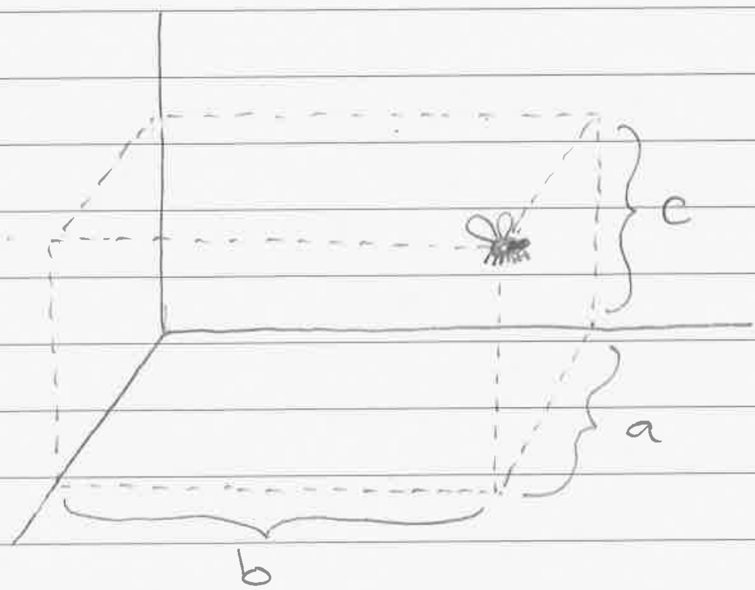
Our modern understanding is based on a revolutionary idea of René Descartes and Pierre de Fermat from the early 1600s.

☆ Revolutionary Idea:

A point is an ordered list of numbers.

What ?!

Apparently Descartes was lying in bed and he saw a fly buzzing in the corner.



He imagined that the fly was at the corner of a rectangular box with dimensions a, b, c.

Descartes realized that the numbers a, b, c (in some fixed order) uniquely specify the position of the fly!

(a, b, c) = the "(Des)cartesian coordinates" of the fly.

In this class we will write the coordinates as a vertical column

$$\vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

and we will call this a vector. By also allowing negative numbers we can uniquely specify the position of any point in space. So we have

point = vector = vertical column of numbers.

The point with all coordinates $= 0$ is a very special point called the origin of the coordinate system.

But this is more than just a notation because it suggests new kinds of things we can do to points.

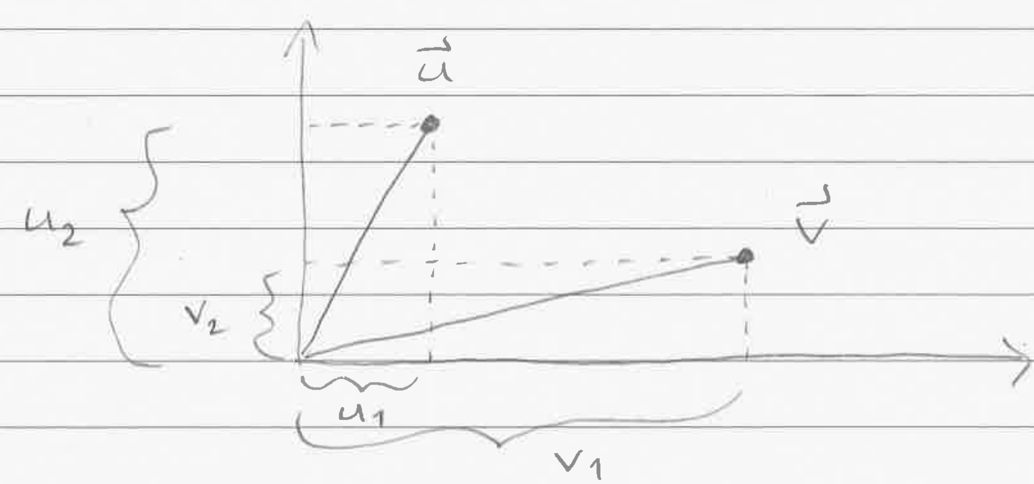
For example, we can "add" them.

Given vectors $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ & $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

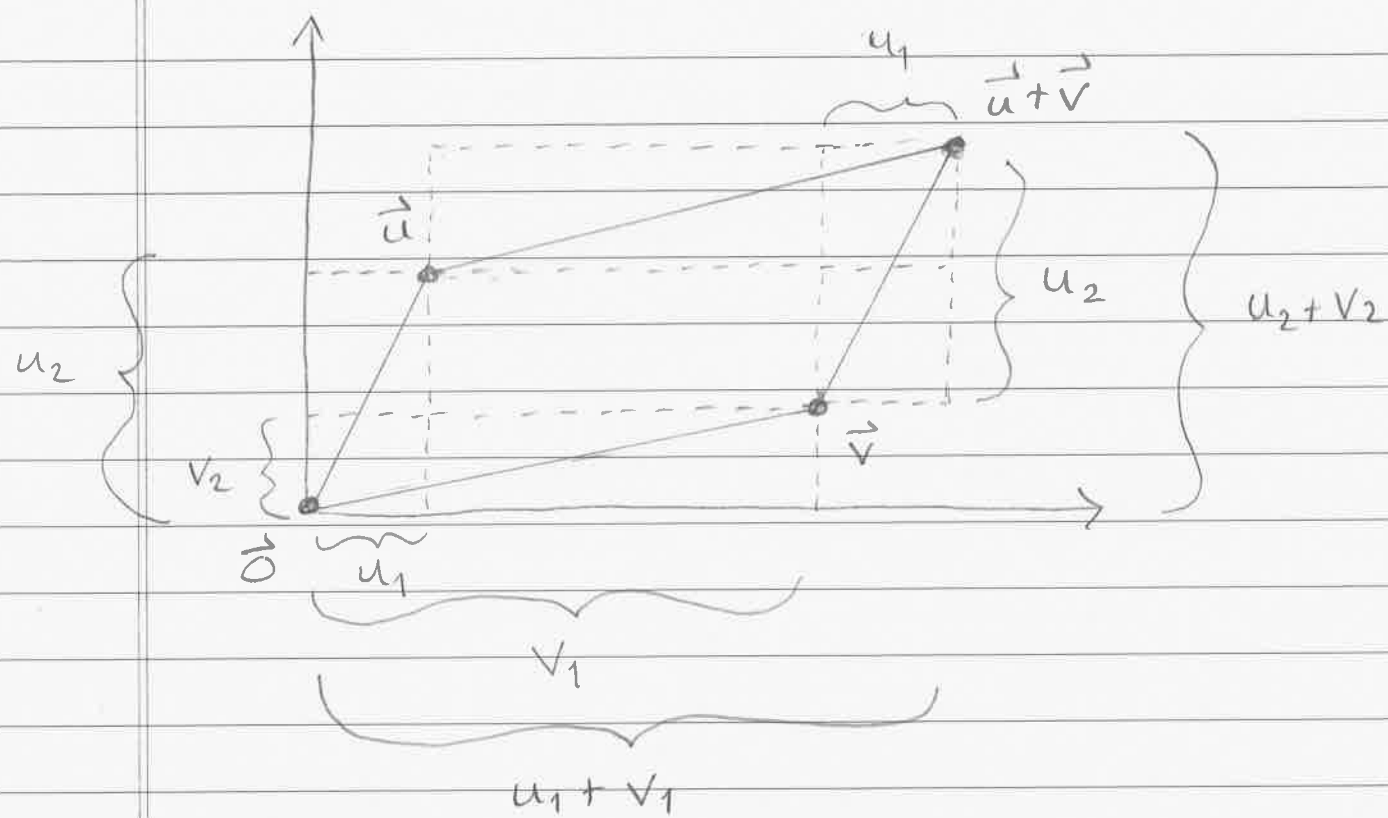
we will define

$$\vec{u} + \vec{v} \underset{\text{definition}}{:=} \begin{pmatrix} u_1 + u_2 \\ v_1 + v_2 \end{pmatrix}.$$

That is, we add two vectors by adding their respective components (or entries). This definition seems obvious in terms of numbers, but what does it mean geometrically? Let's first draw the points $\vec{u}$ & $\vec{v}$.



Then we have



Notice that the points $\vec{0} := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\vec{u}$, $\vec{v}$ and $\vec{u}+\vec{v}$ form the four vertices of a parallelogram. This is called the

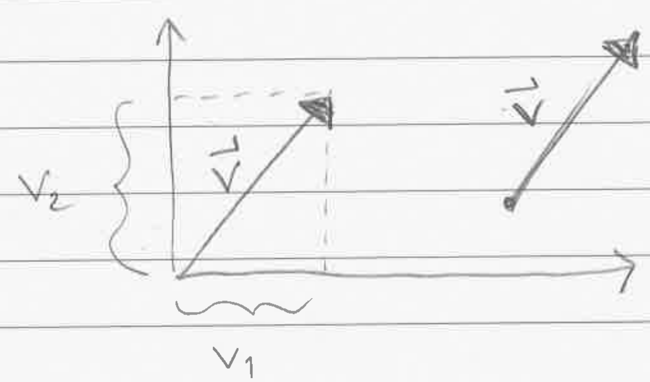
Parallelogram Law

of vector addition.

The parallelogram law suggests a subtle but very useful idea

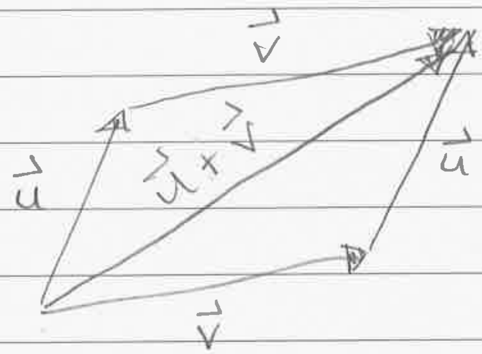
☆ Subtle Idea:

Sometimes we will think of the vector $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ as an <u>arrow</u> with head at the point $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and tail at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.



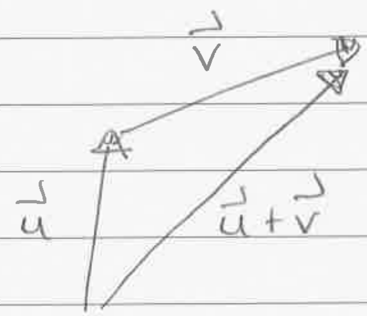
The subtle thing is that we're allowed to pick up the arrow and move it, as long as we don't change its length or direction

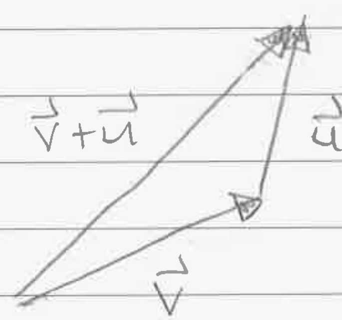
The useful thing is that this makes addition and subtraction of vectors very easy to describe.



Vectors add "head-to-tail". Note from the picture that addition of vectors is <u>commutative</u>; that is, we have

$$\vec{u}+\vec{v} = \vec{v}+\vec{u}$$





Question: How can we use this idea to <u>subtract</u> vectors?

1/13/16

Recall from last time:

A vector is a vertical column of numbers

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Here we say that n is the dimension of the vector $\vec{v}$. Vectors of the same dimension can be added as follows:

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} := \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{pmatrix}$$

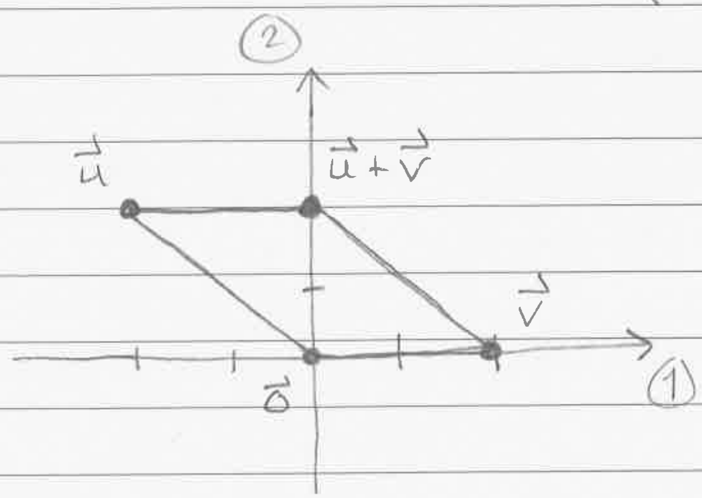
If $n=2$ or $n=3$ then vector addition has a nice geometric interpretation.

☆ Parallelogram Law: Let $\vec{u}$ & $\vec{v}$ be vectors in 2 or 3 dimensions, thought of as <u>points</u> in Cartesian coordinates. Then the four points

$$\vec{0}, \quad \vec{u}, \quad \vec{v}, \quad \vec{u}+\vec{v}$$

are the vertices of a para<u>llelo</u>gram.

Example: Let $\vec{u} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ & $\vec{v} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.



See the parallelogram?

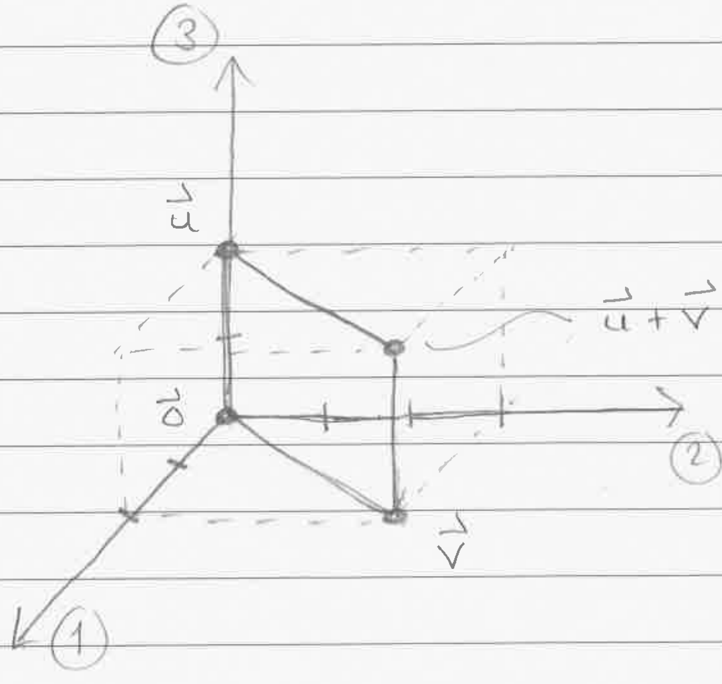
Note that

$$\vec{u} + \vec{v} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2+2 \\ 2+0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

agrees with the picture. //

Example: $\vec{u} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$ & $\vec{v} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$



$$\vec{u} + \vec{v} = \begin{pmatrix} 0+2 \\ 0+3 \\ 2+0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

See the 2D parallelogram living in 3D?
[It's actually a rectangle.] //

The pictures for $n \geq 4$ are harder to draw. :)

There is another important way to think about vector addition

☆ Subtle Idea: Sometimes we will think of the vector

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

as an <u>arrow</u> with tail at $(0, 0, \ldots, 0)$ and head at $(v_1, v_2, \ldots, v_n)$.

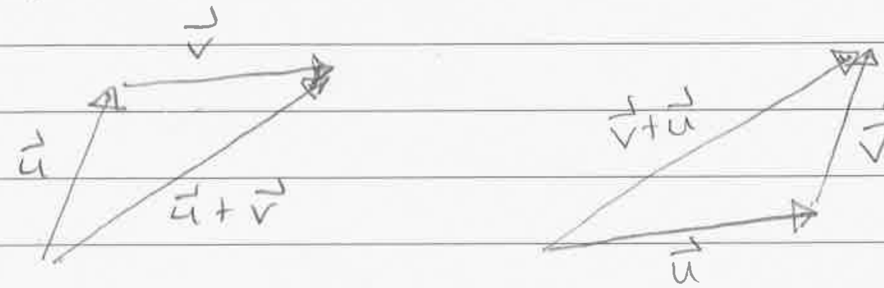
[Remark: Sometimes I will write a vector as a list with commas. This is just to save space; it's still a vertical column in my mind.]

The subtle thing is that we can pick up the arrow and move it around, as long as we don't change its <u>length</u> or <u>direction</u>.

In this languge, we can add vectors "head-to-tail"

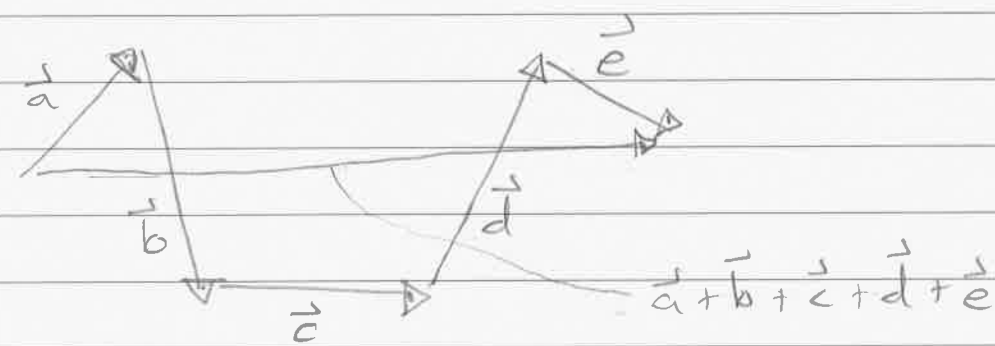
Examples:

(i)



Note that $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ because the are both the diagonal of the same parallelogram.

(ii) It works with multiple vectors also:



Adding the vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}$ in any order [there are 120 ways to do this] gives the same resut!

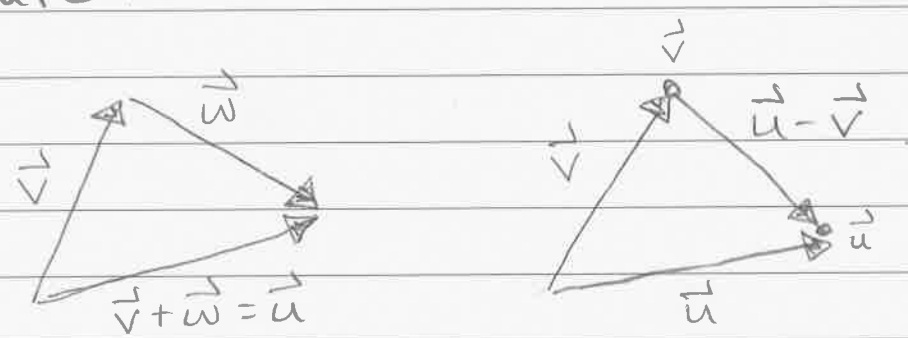
This language also suggests how to <u>subtract</u> vectors.

Consider two vectors $\vec{u}$ & $\vec{v}$ of the same dimension. Which vector $\vec{w}$ deserves to be called "$\vec{u}-\vec{v}$" ?

If $\vec{w} =$ "$\vec{u}-\vec{v}$" then we should have

$$\vec{v}+\vec{w}=\vec{u}, \quad \text{yes ?}$$

Picture!

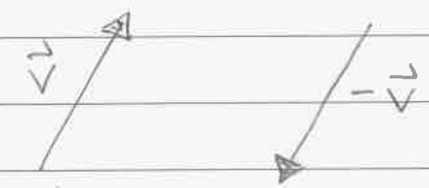


So we could define $\vec{u}-\vec{v}$ as the arrow with tail at the point $\vec{v}$ and head at the point $\vec{u}$. One can check that this picture agrees with the algebraic rule

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} := \begin{pmatrix} u_1-v_1 \\ u_2-v_2 \\ \vdots \\ u_n-v_n \end{pmatrix}$$

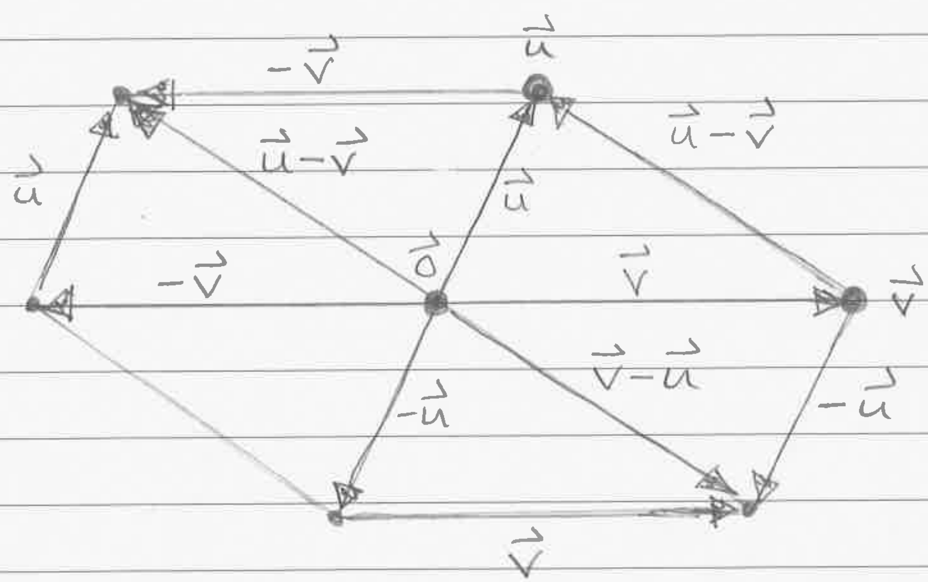
But there is yet another way to think about subtraction of vectors:

Given an arrow $\vec{v}$ we will define the arrow "$-\vec{v}$" with the head and tail switched:



Then we have $\vec{u}-\vec{v} = \vec{u}+(-\vec{v})$.

Picture:



Everything fits together nicely. ☺

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In summary, there are two ways to think about vectors

(1) Algebra of lists of numbers.
(2) Geometry of arrows.

The interaction between these two perspectives is what gives Linear Algebra its power.

We have seen that there are two ways to think about the vector

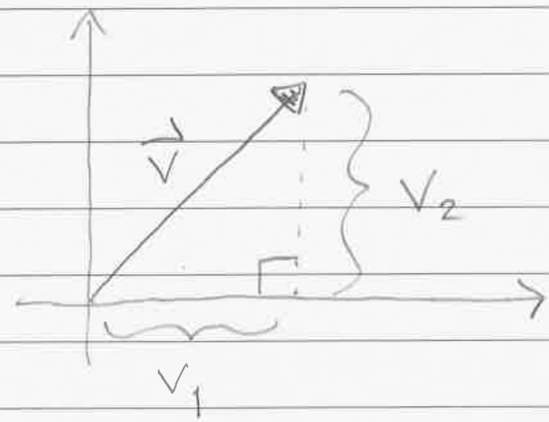
$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

(1) The Cartesian coordinates of a point in "n-dimensional space".

(2) The arrow with tail at $(0, 0, \ldots, 0)$ and head at $(v_1, v_2, \ldots, v_n)$. We are allowed to move the arrow as long as we don't change its length or direction.

But what *is* the length of this vector?

Example ($n=2$): Let $\vec{v} = (v_1, v_2)$.



Let $\|\vec{v}\|$ denote the length of $\vec{v}$. Since the coordinate axes are perpendicular we have a right triangle and we can use the Pythagorean Theorem to get

$$\|\vec{v}\|^2 = v_1^2 + v_2^2$$

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}.$$

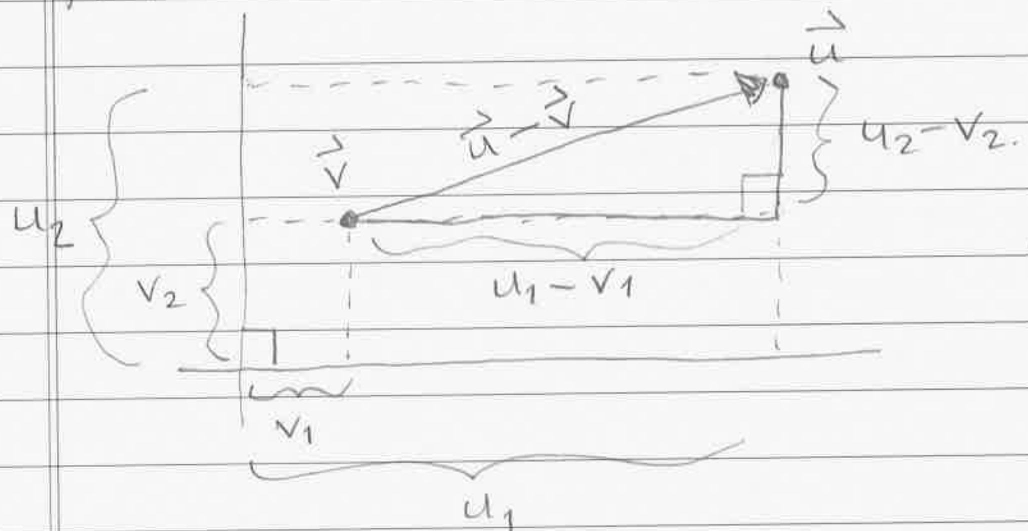
More generally, consider any two vectors $\vec{u} = (u_1, u_2)$ & $\vec{v} = (v_1, v_2)$ and their difference

$$\vec{u} - \vec{v} = (u_1 - v_1, u_2 - v_2).$$

The length of the arrow $\vec{u} - \vec{v}$ is the same as the distance between the points $\vec{u}$ & $\vec{v}$, so we have

(*) $$\text{dist}(\vec{u}, \vec{v}) = \| \vec{u} - \vec{v} \| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$$

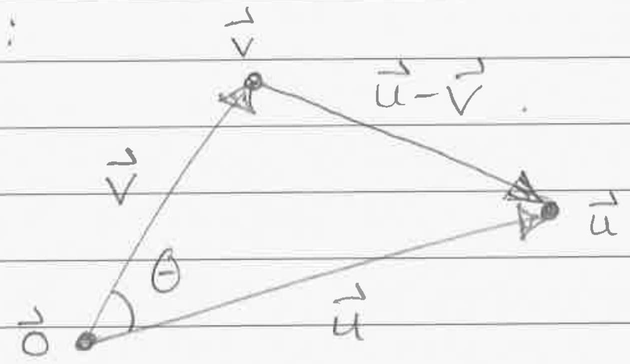
Picture:



So formula (*) is just the Pythagorean Theorem again.

But remember that there is another useful way to think of the arrow $\vec{u} - \vec{v}$: as the third side of a triangle with arrows $\vec{u}$ & $\vec{v}$.

Picture:



Let θ be the angle between $\vec{u}$ & $\vec{v}$ when both of their tails are at $(0,0)$. Maybe you remember a fact about triangles called the Law of Cosines; it says that

$$(**) \qquad \|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\cdot\|\vec{v}\|\cos\theta .$$

But now we have two expressions for the length/distance $\|\vec{u}-\vec{v}\|$ and a very interesting thing happens if we compare them.

From $(*)$ we have

$$\begin{aligned}
\|\vec{u}-\vec{v}\|^2 &= (u_1-v_1)^2 + (u_2-v_2)^2 \\
&= (u_1^2 - 2u_1v_1 + v_1^2) + (u_2^2 - 2u_2v_2 + v_2^2) \\
&= (u_1^2+u_2^2) + (v_1^2+v_2^2) - 2(u_1v_1+u_2v_2). \\
&= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2(u_1v_1+u_2v_2).
\end{aligned}$$

Then equating the two expressions for $\|\vec{u}-\vec{v}\|^2$ gives

$$\cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} - 2(u_1v_1+u_2v_2) = \cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} - 2\|\vec{u}\|\cdot\|\vec{v}\|\cos\theta$$

$$-2(u_1v_1+u_2v_2) = -2\|\vec{u}\|\cdot\|\vec{v}\|\cos\theta$$

$$\boxed{u_1v_1+u_2v_2 = \|\vec{u}\|\cdot\|\vec{v}\|\cos\theta}$$

This is a very funny formula. The quantity on the left is new to us so we need to give it a name.

☆ Definition: Given two n-dimensional vectors $\vec{u}=(u_1, \ , u_n)$ & $\vec{v}=(v_1,\dots,v_n)$ we will define their dot product

$$\vec{u}\bullet\vec{v} := u_1v_1+u_2v_2+\cdots+u_nv_n.$$

[Note that the dot product of two vectors is a number, not a vector.]

We have seen that the dot product in two dimensions has the geometric interpretation

$$\underbrace{\vec{u} \bullet \vec{v}}_{\text{algebra}} = \underbrace{\|\vec{u}\| \cdot \|\vec{v}\| \cos\theta}_{\text{geometry}},$$

where θ is the angle between the arrows $\vec{u}$ & $\vec{v}$ when their tails are at $\vec{0}$. As a special case we have

$$\begin{aligned} \vec{u} \bullet \vec{u} &= \|\vec{u}\| \cdot \|\vec{u}\| \cos 0 \\ &= \|\vec{u}\|^2 \cdot 1 \\ &= \|\vec{u}\|^2 . \end{aligned}$$

So we can also express the <u>length</u> of a vector in terms of the dot product

$$\|\vec{u}\| = \sqrt{\vec{u} \bullet \vec{u}} .$$

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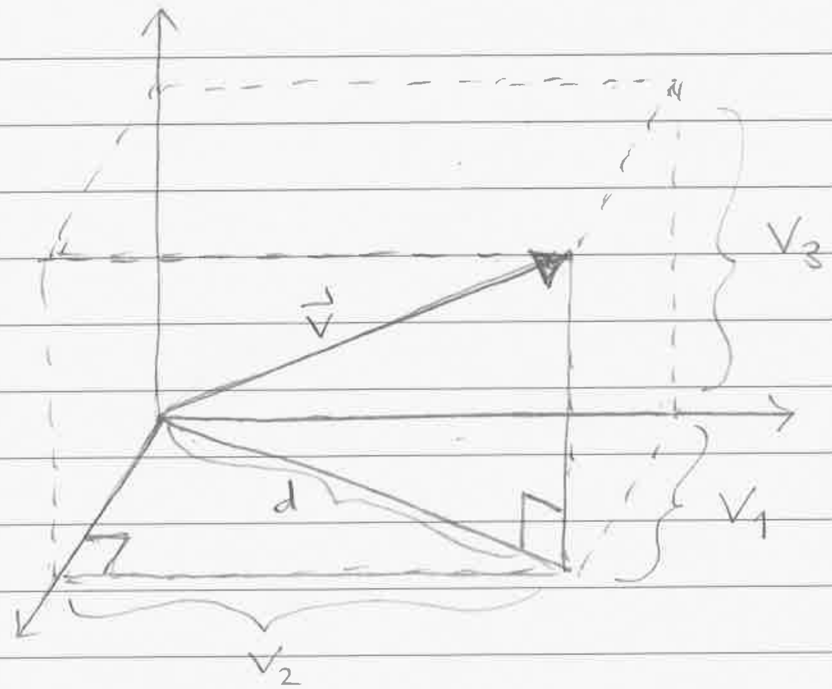
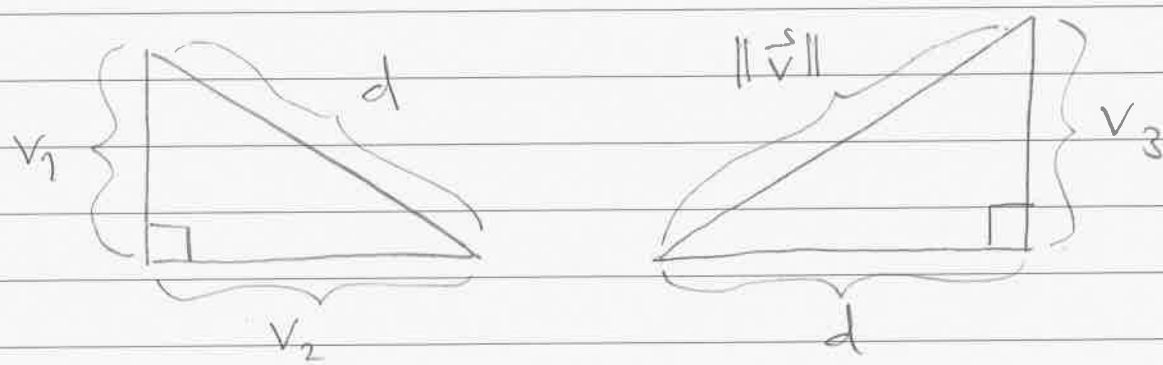
Q: Do the same formulas hold in three-dimensional space?

Consider the vector $\vec{v} = (v_1, v_2, v_3)$.
It is the diagonal of a rectangular box:



To compute the length $\|\vec{v}\|$ we will use two right triangles:



Applying the Pythagorean Theorem to both triangles gives

$$d^2 = v_1^2 + v_2^2 \quad \& \quad \|\vec{v}\|^2 = d^2 + v_3^2$$

and hence

$$\begin{aligned} \|\vec{v}\|^2 &= d^2 + v_3^2 \\ &= (v_1^2 + v_2)^2 + v_3^2 \\ &= v_1^2 + v_2^2 + v_3^2 . \end{aligned}$$

We conclude that

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} .$$

///

And this fact means that all of our formulas from 2D are still true in 3D.

Q: How about higher dimensions?
If $\vec{v} = (v_1, v_2, v_3, v_4)$, is it <u>true</u> that

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2} \quad ?$$

A: Sure, why not?

Recall: Given two n-dimensional vectors

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \quad \& \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix},$$

their dot product is the number defined by

$$\vec{u} \cdot \vec{v} := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n .$$

" vector • vector = number "

The length of a vector satisfies

$$\|\vec{v}\|^2 = \vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + \cdots + v_n^2$$

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} .$$

When $n = 2$ or 3, we saw last time that this formula comes from the Pythagorean Theorem. For $n \geq 4$ we might as well use the formula $\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$ as the definition of "length".

Thinking Problem: What does $\|\vec{v}\|$ mean when $n = 1$?

Answer: A 1-dimensional vector is just a number, $\vec{v} = (v_1)$. In this case we might as well use the notation

$$\vec{v} = (v).$$

Then the formula for length gives

$$\|\vec{v}\| = \sqrt{v^2} = |v|$$

Thus the "length" of a vector generalizes the "absolute value" of a number. ///

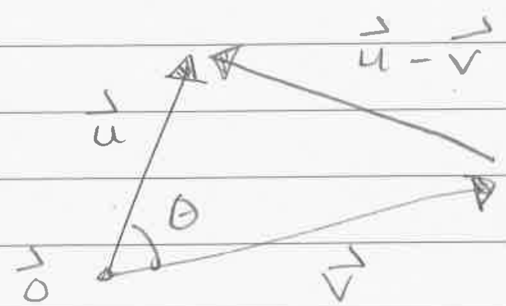
Based on this, the distance between two points $\vec{u}$ & $\vec{v}$ in n-dimensional space is defined by

$$\mathrm{dist}(\vec{u},\vec{v}) := \|\vec{u}-\vec{v}\| = \sqrt{(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})}\,.$$

Using a purely algebraic argument gives

$$(*)\quad \begin{aligned}\|\vec{u}-\vec{v}\|^2 &= (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})\\ &= (\vec{u}-\vec{v})\cdot\vec{u} - (\vec{u}-\vec{v})\cdot\vec{v}\\ &= \vec{u}\cdot\vec{u} - \vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}\\ &= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - 2(\vec{u}\cdot\vec{v})\\ &= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2(\vec{u}\cdot\vec{v})\end{aligned}$$

On the other hand, we can think of the vectors $\vec{u}$, $\vec{v}$, $\vec{u}-\vec{v}$ as forming a 2D triangle in n-dimensional space:



Then the Law of Cosines says that

$$(**)\quad \|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\cdot\|\vec{v}\|\cos\theta$$

Finally, equating the expressions $(*)$ & $(**)$ for the number $\|\vec{u}-\vec{v}\|^2$ gives (after some simplification)

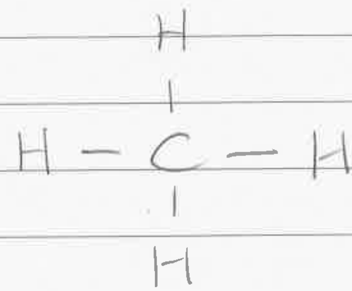
$$\boxed{\vec{u} \circ \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos\theta}$$

Thus we have a geometric interpretation for the dot product that holds in any dimension.

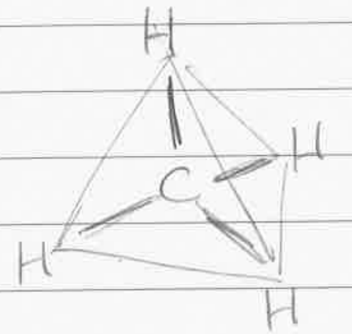
Thinking Problem: How should you interpret the boxed formula when $n=1$?

Application to Chemistry:

A molecule of methone consists of one carbon atom surrounded by four hydrogen atoms:



But it doesn't look like this in real life; for symmetry reasons it has the shape of a "regular tetrahedron" with H's at the vertices and C at the center.



Chemistry textbooks often say that the angle between any two hydrogen atoms is

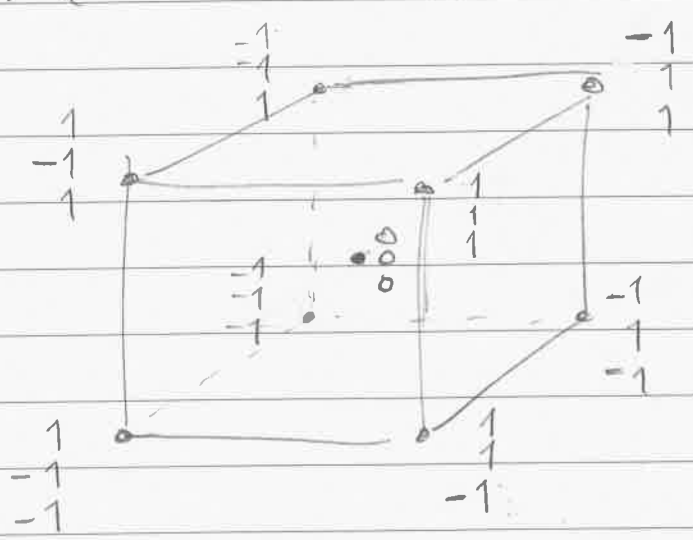
$$109.5^\circ.$$

But <u>why</u> is this? Let's use the dot product.

First we need a coordinate system.

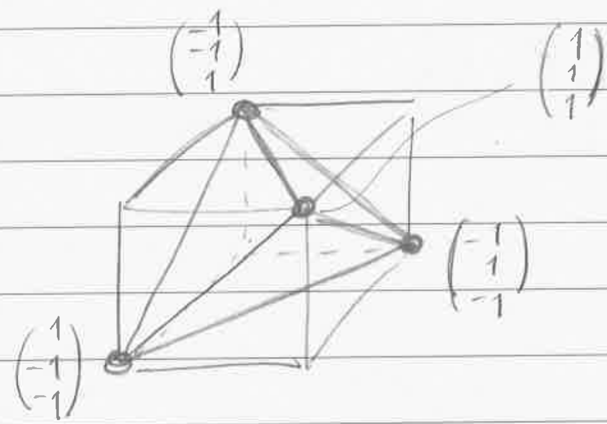
Let's place C at the origin $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Now let's draw the cube whose coordinates are ± 1 (which is centered at $\vec{0}$):

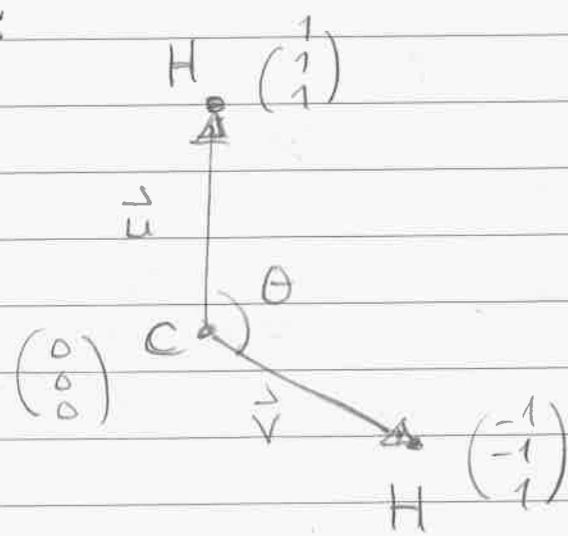


[Compare to HW1(a).] The cube has 8 vertices. It turns out that if we choose 4 "alternating" vertices then they form the vertices of a regular tetrahedron centered at $\vec{0}$.

There are two choices; here's one:



Finally, let's compute the angle between any two vertices. There are six choices; here's one:



We have

$$\|\vec{u}\|^2 = \vec{u} \circ \vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \circ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1^2 + 1^2 + 1^2 = 3$$

$$\|\vec{v}\|^2 = \vec{v} \circ \vec{v} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \circ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = (-1)^2 + (-1)^2 + 1^2 = 3.$$

$$\vec{u} \bullet \vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \circ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = -1 - 1 + 1 = -1.$$

Then our boxed formula gives

$$\vec{u} \circ \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos\theta$$

$$-1 = \sqrt{3} \cdot \sqrt{3} \cos\theta.$$

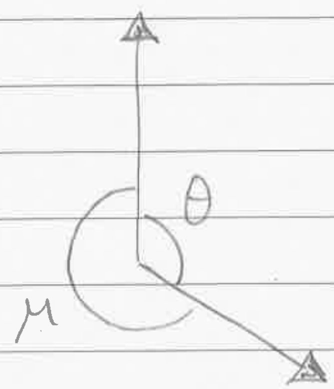
$$-1 = 3 \cos\theta$$

$$-\frac{1}{3} = \cos\theta,$$

and hence

$$\theta = \arccos\left(-\frac{1}{3}\right) \approx 109.47122^\circ.$$

Actually, there are two different angles with cosine $-1/3$. But there are also two different angles between the vectors:



Note that $\cos\mu = \cos\theta = -1/3$ where

$$\theta \approx 109.5^\circ \quad \text{and} \quad \mu \approx 360 - 109.5 = 250.5^\circ$$

So everything works out nicely :).

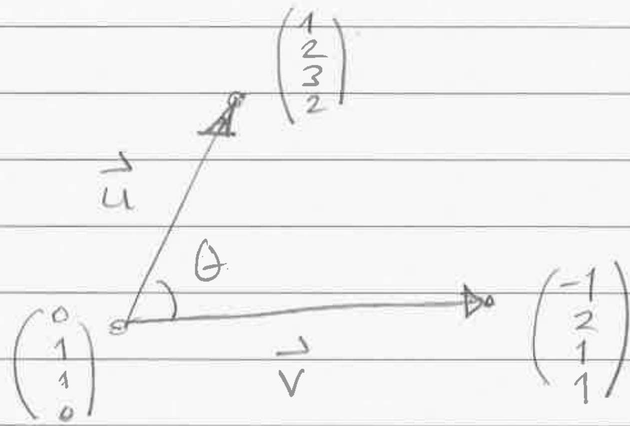
Therefore it is reasonable to leave our answer in the form

$$\cos\theta = -1/3.$$

//

Today: HW1 Discussion.

Problem 1′: Consider the following (not accurate) picture in 4-dimensional space.



Use the dot product to compute the angle.

Solution: Our formula says

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos\theta$$

but to compute $\vec{u} \cdot \vec{v}$ we first need to move the tails of $\vec{u}$ & $\vec{v}$ to the origin. The trick for doing this is

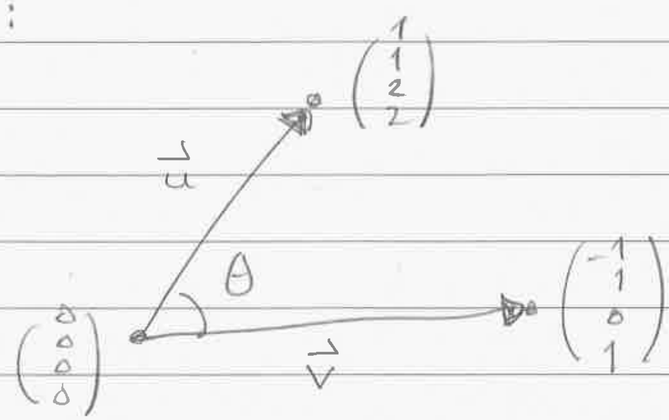
"head − tail".

We have

$$\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Picture:



Then we have

$$\vec{u} \cdot \vec{v} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= 1 \cdot (-1) + 1 \cdot 1 + 2 \cdot 0 + 2 \cdot 1$$

$$= -1 + 1 + 0 + 2 = 2 ,$$

$$\|\vec{u}\| = \sqrt{1^2+1^2+2^2+2^2} = \sqrt{10} = 5\sqrt{2}$$

$$\|\vec{v}\| = \sqrt{(-1)^2+1^2+0^2+1^2} = \sqrt{3}.$$

Plugging into the formula gives

$$\vec{u}\circ\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\| \cos\theta$$

$$2 = 5\sqrt{2}\cdot\sqrt{3} \cos\theta$$

$$\cos\theta = \frac{2}{5\sqrt{6}}$$

$$\theta \approx 68.6^\circ$$

///

Problem 2′: Draw the points and Lines from Problem 2 (a), (b), (c).

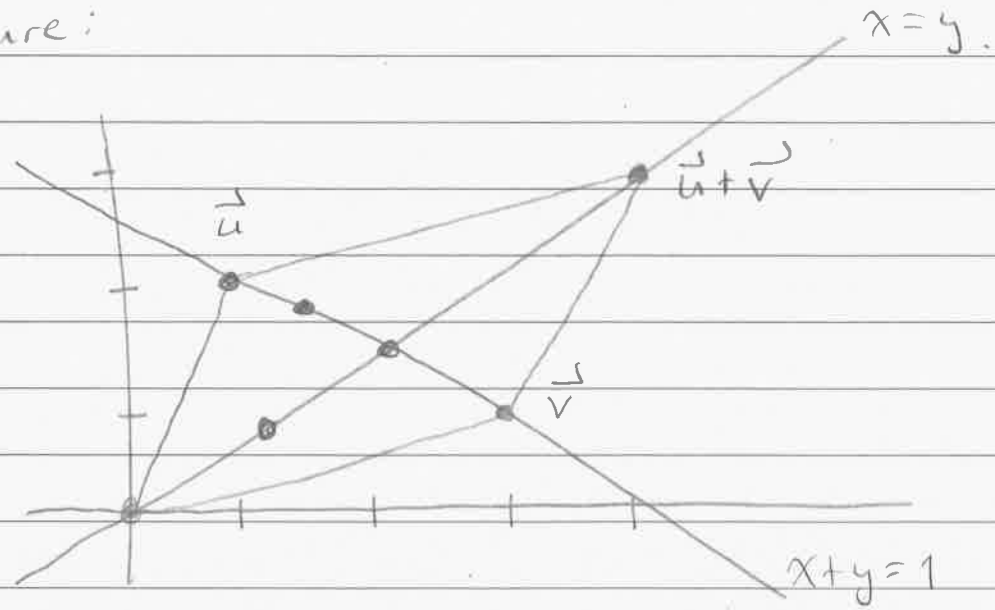
$$\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \; \& \; \vec{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Points : $\vec{u}$, $\vec{v}$, $\frac{1}{2}\vec{u}+\frac{1}{2}\vec{v}$, $\frac{3}{4}\vec{u}+\frac{1}{4}\vec{v}$, $\frac{1}{4}\vec{u}+\frac{1}{4}\vec{v}$, $\vec{u}+\vec{v}$

Lines : $x\vec{u}+y\vec{v}$ where $x+y=1$

$x\vec{u}+y\vec{v}$ where $x=y$.

Picture:



[Remark: If $\vec{a}$ & $\vec{b}$ are any two points their midpoint / average is given by

$$\frac{\vec{a}+\vec{b}}{2} = \frac{1}{2}(\vec{a}+\vec{b}) = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}.$$

Thus, for example, the midpoint of $\vec{u}$ and $\frac{1}{2}\vec{u}+\frac{1}{2}\vec{v}$ is given by

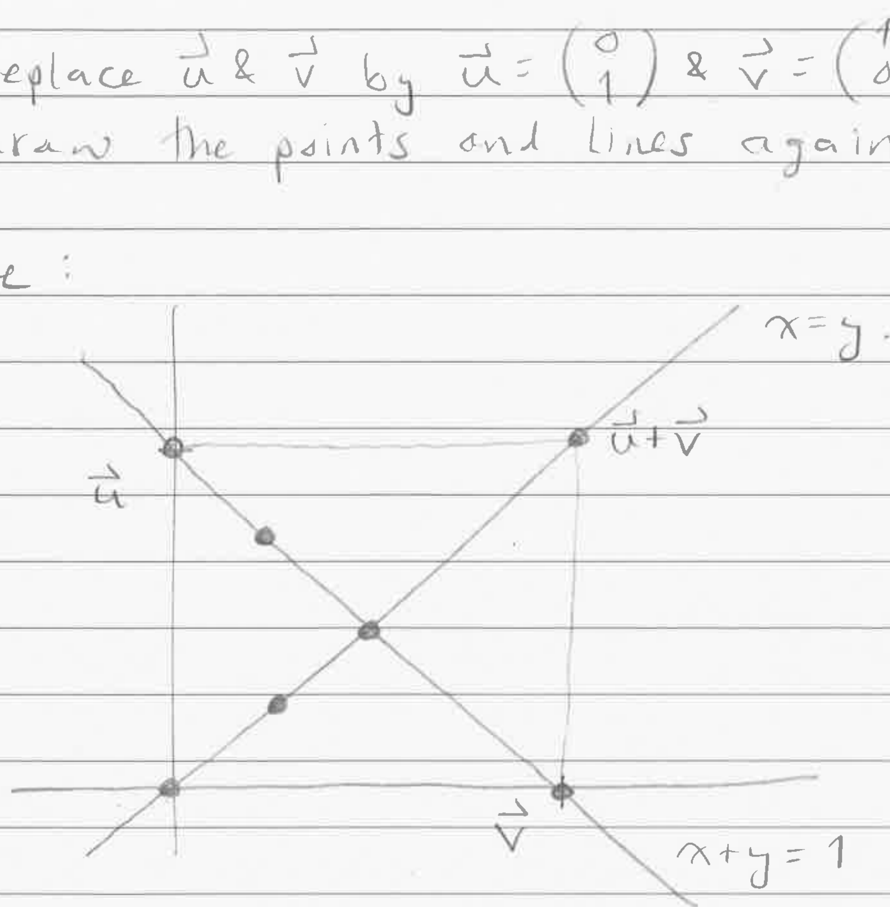
$$\frac{1}{2}\vec{u} + \frac{1}{2}\left(\frac{1}{2}\vec{u}+\frac{1}{2}\vec{v}\right) = \frac{1}{2}\vec{u} + \frac{1}{4}\vec{u} + \frac{1}{4}\vec{v}$$

$$= \frac{3}{4}\vec{u} + \frac{1}{4}\vec{v},$$

as seen in the picture.]

Now replace $\vec{u}$ & $\vec{v}$ by $\vec{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ & $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and draw the points and lines again.

Picture:



Does the picture look similar? Good.

All we really did is "change the coordinate system". The standard Cartesian coordinates in the plane are defined by

$$\vec{e}_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \& \quad \vec{e}_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

so that the point $\begin{pmatrix} x \\ y \end{pmatrix}$ can be expressed as

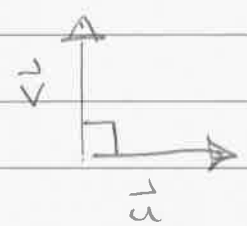
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = x\vec{e}_1 + y\vec{e}_2 .$$

//

Problem 4': In Problem 4 I gave you a coordinate system $\vec{u}$ & $\vec{v}$ without telling you much about it. All we know is

$$\vec{u}\circ\vec{u} = \vec{v}\circ\vec{v} = 1 \quad \& \quad \vec{u}\circ\vec{v} = 0.$$

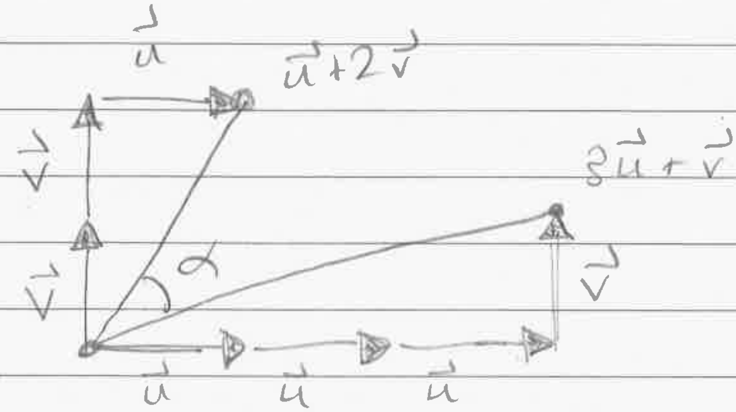
But this is enough to draw a picture:



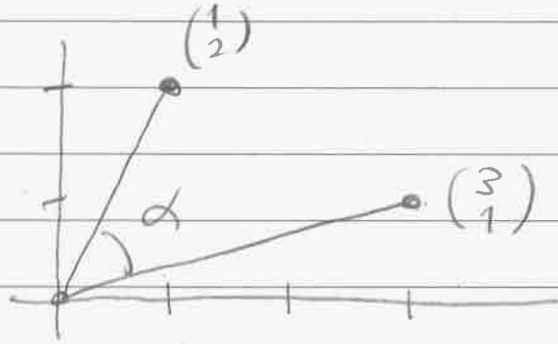
[The vectors are perpendicular because

$$\begin{aligned} \vec{u}\circ\vec{v} &= \|\vec{u}\|\,\|\vec{v}\|\cos\theta \\ 0 &= 1\cdot 1\cos\theta \\ 0 &= \cos\theta \\ 90^\circ &= \theta. \end{aligned}$$
]

Then I asked you to compute the angle between $\vec{u}+2\vec{v}$ & $3\vec{u}+\vec{v}$.



As you see from the picture, this is pretty much the same as computing the angle between $\binom{1}{2}$ & $\binom{3}{1}$ in the 2D plane.



and if you compute both you'll get the same answer

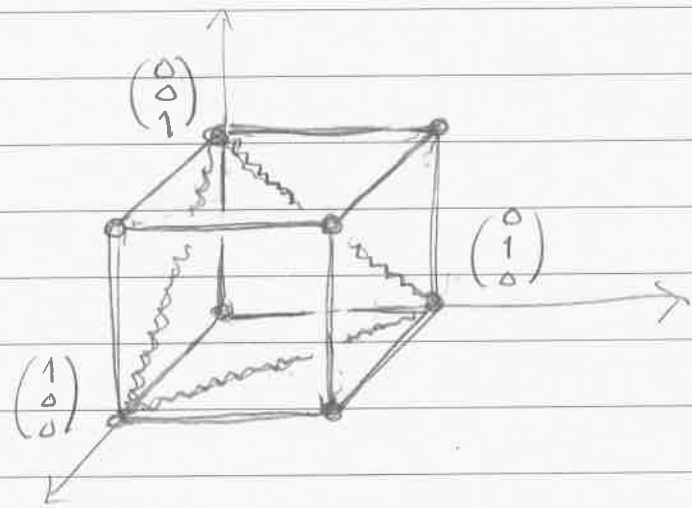
$$\alpha = 45^\circ.$$

Not a surprise.

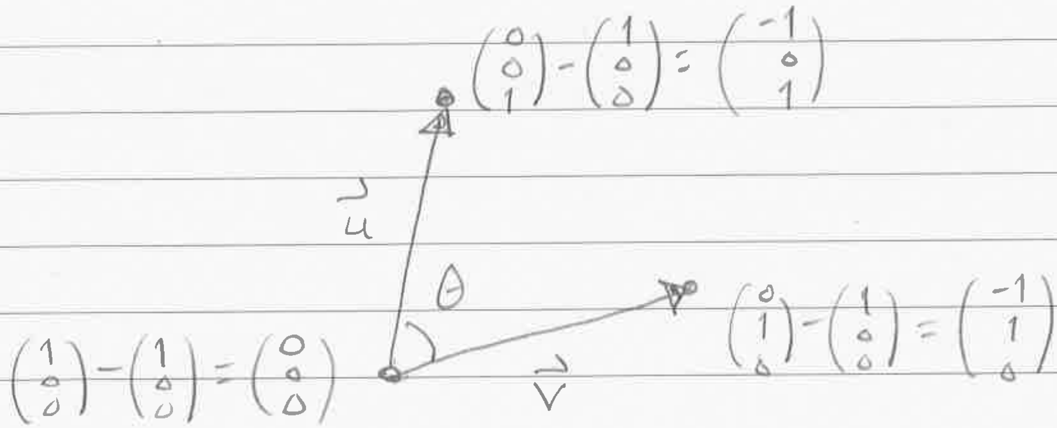
Old Homework Solutions

Problem 1 :

(a)



(b) The triangle is shown above with the squiggly lines. To compute the angle between two edges we should move the corresponding vectors to the origin. Here's one case:



To compute the angle we use the dot product.

$$\cos\theta = \vec{u}\cdot\vec{v} \Big/ \left(\|\vec{u}\|\cdot\|\vec{v}\| \right).$$

$$= \begin{pmatrix}-1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\0\end{pmatrix} \Bigg/ \left(\left\|\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\| \cdot \left\|\begin{pmatrix}-1\\1\\0\end{pmatrix}\right\| \right)$$

$$= \left((-1)^2+0+0\right) \Big/ \left(\sqrt{(-1)^2+0^2+1^2}\cdot\sqrt{(-1)^2+1^2+0^2} \right)$$

$$= 1 \Big/ \left(\sqrt{2}\cdot\sqrt{2}\right) = 1/2.$$

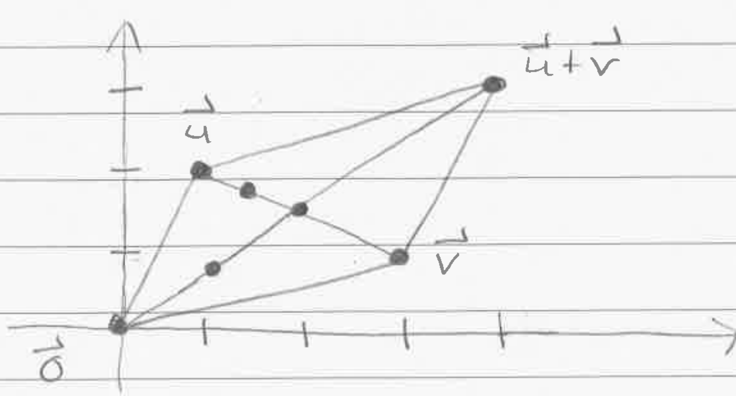
We conclude that

$$\theta = \arccos\left(1/2\right) = \pi/3 \quad (\text{or } 60^\circ).$$

It turns out that all three angles are the same [the triangle is equilateral]. ///

Problem 2:

(a)



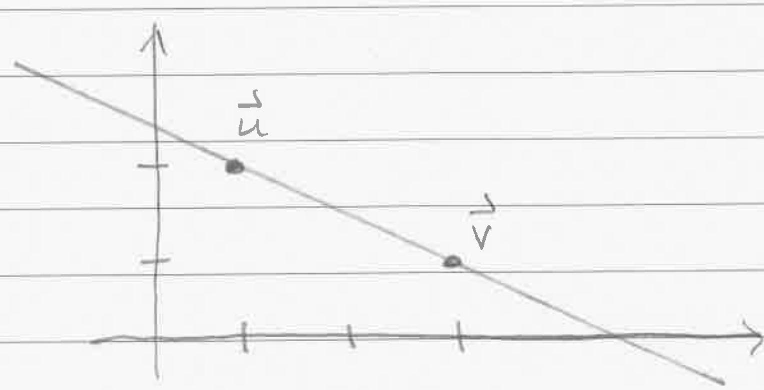
Note that

- $\frac{1}{2}\vec{u} + \frac{1}{2}\vec{v}$ is the midpoint of $\vec{u}$ & $\vec{v}$.
- $\frac{3}{4}\vec{u} + \frac{1}{2}\vec{v}$ is the midpoint of $\vec{u}$ & $\frac{1}{2}\vec{u} + \frac{1}{2}\vec{v}$.
- $\frac{1}{4}\vec{u} + \frac{1}{4}\vec{v}$ is the midpoint of $\vec{0}$ & $\frac{1}{2}\vec{u} + \frac{1}{2}\vec{v}$.

[In general, the midpoint of points $\vec{a}$ & $\vec{b}$ is

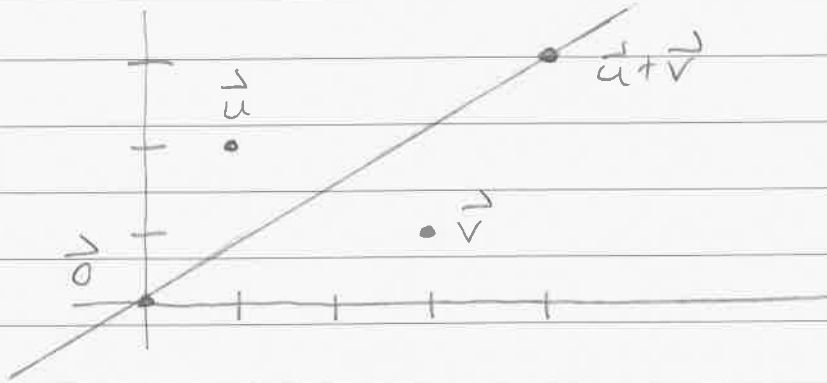
$$\frac{1}{2}(\vec{a} + \vec{b}) = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}.$$]

(b) The line $a\vec{u} + b\vec{v}$ where $a+b=0$ contains the point $\vec{u}$ (when $a=1$ & $b=0$) and the point $\vec{v}$ (when $a=0$ & $b=1$). Hence it looks like this:



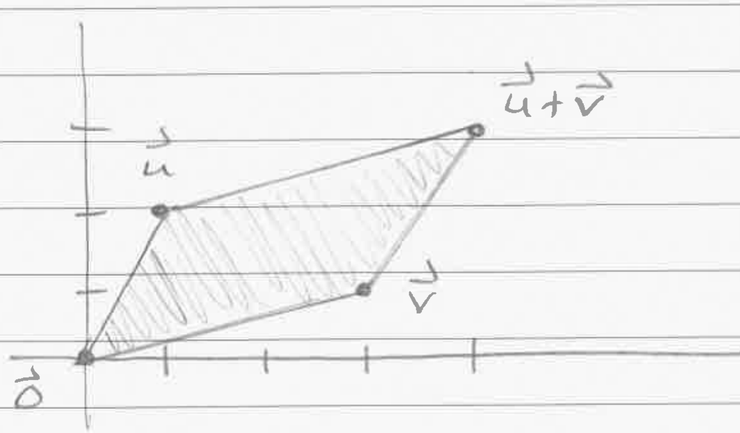
[You might try computing the equation of this line.]

(c) The line $a\vec{u} + a\vec{v}$ contains the point $\vec{0}$ (when $a = 0$) and the point $\vec{u}+\vec{v}$ (when $a = 1$). Hence it looks like this:

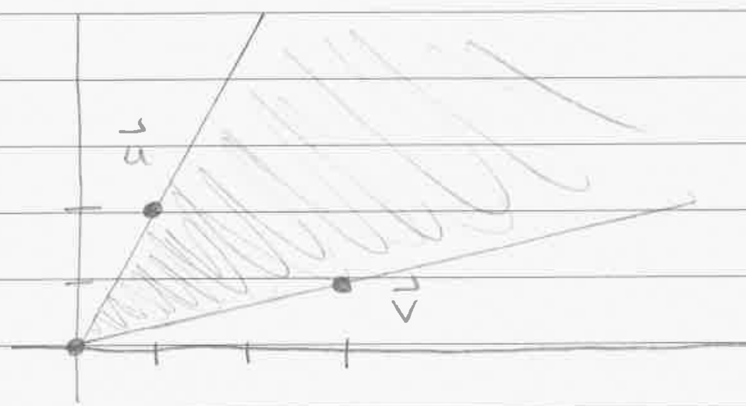


[The equation of this line is . . .]

(d) The region $a\vec{u} + b\vec{v}$ when $0 \leq a \leq 1$ and $0 \leq b \leq 1$ is the filled parallelogram with vertices $\vec{0}$, $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$:



(d) The region $a\vec{u} + b\vec{v}$ with $0 \le a$ & $0 \le b$. We could call this region a "two-dimensional cone". It looks like this:



Problem 3: Let $\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ & $\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$.

Then for all numbers a we have

$$\vec{u} \bullet (\vec{v} + a\vec{w}) = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \bullet \begin{pmatrix} v_1 + aw_1 \\ \vdots \\ v_n + aw_n \end{pmatrix}$$

$$= u_1(v_1 + aw_1) + \cdots + u_n(v_n + aw_n)$$

$$= (u_1v_1 + au_1w_1) + \cdots + (u_nv_n + av_nw_n)$$

$$= (u_1v_1 + \cdots + u_nv_n) + a(u_1w_1 + \cdots + u_nw_n)$$

$$= \vec{u} \circ \vec{v} + a\, \vec{u} \circ \vec{w}.$$

///

Problem 4: Let $\vec{u}$ & $\vec{v}$ be two vectors in n-dimensional space such that

$$\|\vec{u}\| = \|\vec{v}\| = 1.$$

(a) Then we have

$$\vec{u} \circ (-\vec{u}) = -(\vec{u} \circ \vec{u}) = -\|\vec{u}\|^2 = -1^2 = -1.$$

$$\begin{aligned}(\vec{u}+\vec{v}) \circ (\vec{u}-\vec{v}) &= (\vec{u}+\vec{v}) \circ \vec{u} - (\vec{u}+\vec{v}) \circ \vec{v} \\ &= \vec{u} \circ \vec{u} + \cancel{\vec{v} \circ \vec{u}} - \cancel{\vec{u} \circ \vec{v}} - \vec{v} \circ \vec{v} \\ &= \vec{u} \circ \vec{u} - \vec{v} \circ \vec{v} \\ &= \|\vec{u}\|^2 - \|\vec{v}\|^2 \\ &= 1^2 - 1^2 \qquad = 0.\end{aligned}$$

$$\begin{aligned}(\vec{u}-2\vec{v}) \circ (\vec{u}+2\vec{v}) &= (\vec{u}-2\vec{v}) \circ \vec{u} + 2(\vec{u}-2\vec{v}) \circ \vec{v} \\ &= \vec{u} \circ \vec{u} - \cancel{2\vec{v} \circ \vec{u}} + \cancel{2\vec{u} \circ \vec{v}} - 4\vec{v} \circ \vec{v} \\ &= \vec{u} \circ \vec{u} - 4\vec{v} \circ \vec{v} \\ &= \|\vec{u}\|^2 - 4\|\vec{v}\|^2 \\ &= 1^2 - 4 \cdot 1^2 \qquad = -3.\end{aligned}$$

(b) Now let's also assume that $\vec{u} \circ \vec{v} = 0$ and consider the vectors

$$\vec{a} := \vec{u} + 2\vec{v} \quad \& \quad \vec{b} := 3\vec{u} + \vec{v}.$$

Let θ be the angle between $\vec{a}$ & $\vec{b}$ so that

$$\vec{a} \circ \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cos\theta .$$

To compute θ we first need to know $\vec{a} \circ \vec{b}$, $\|\vec{a}\|$, and $\|\vec{b}\|$.

$$\begin{aligned}
\vec{a} \circ \vec{b} &= (\vec{u} + 2\vec{v}) \circ (3\vec{u} + \vec{v}) \\
&= 3\vec{u} \circ \vec{u} + \vec{u} \circ \vec{v} + 6\vec{v} \circ \vec{u} + 2\vec{v} \circ \vec{v} \\
&= 3\vec{u} \circ \vec{u} + 7\vec{u} \circ \vec{v} + 2\vec{v} \circ \vec{v} . \\
&= 3(1) + 7(0) + 2(1) \quad = \quad 5
\end{aligned}$$

$$\begin{aligned}
\|\vec{a}\|^2 &= \vec{a} \circ \vec{a} \\
&= (\vec{u} + 2\vec{v}) \circ (\vec{u} + 2\vec{v}) \\
&= \vec{u} \circ \vec{u} + 2\vec{u} \circ \vec{v} + 2\vec{v} \circ \vec{u} + 4\vec{v} \circ \vec{v} \\
&= \vec{u} \circ \vec{u} + 4\vec{u} \circ \vec{v} + 4\vec{v} \circ \vec{v} \\
&= (1) + 4(0) + 4(1) \quad = \quad 5
\end{aligned}$$

$$\begin{aligned}
\|\vec{b}\|^2 &= \vec{b} \circ \vec{b} \\
&= (3\vec{u} + \vec{v}) \circ (3\vec{u} + \vec{v}) \\
&= 9\vec{u} \circ \vec{u} + 3\vec{u} \circ \vec{v} + 3\vec{v} \circ \vec{u} + \vec{v} \circ \vec{v} \\
&= 9\vec{u} \circ \vec{u} + 6\vec{u} \circ \vec{v} + \vec{v} \circ \vec{v} \\
&= 9(1) + 6(0) + (1) \quad = \quad 10.
\end{aligned}$$

We conclude that

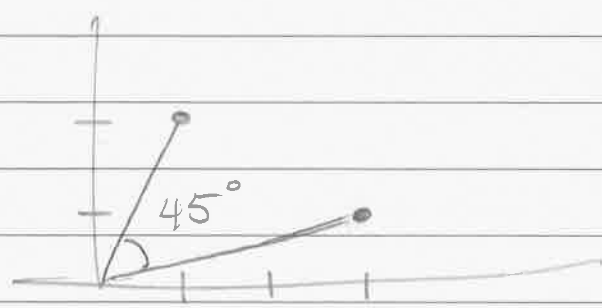
$$\vec{a} \circ \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cos\theta$$
$$5 = \sqrt{5} \cdot \sqrt{10} \cos\theta$$

$$\cancel{5} = \cancel{5}\sqrt{2} \cos\theta$$

$$\cos\theta = 1/\sqrt{2}$$

$$\theta = 45^\circ \quad (\text{or } 315^\circ).$$

[Remark: We get the same result by computing the angle between the vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ in 2D.



I wonder why that might be . . .]