Intro to Linear Algebra
Quiz 5

Summer 2017
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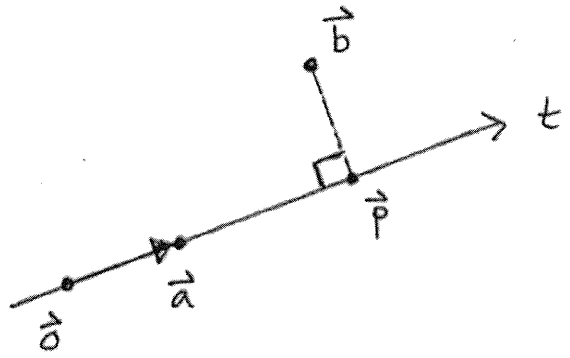
Problem 1. The following picture shows the projection of $\vec{b}$ onto the line through $\vec{a}$:



(a) Write down an equation saying that $\vec{b} - \vec{p}$ is perpendicular to $\vec{a}$.

$$\vec{a}^T(\vec{b} - \vec{p}) = 0$$

(b) The projection has the form $\vec{p} = t\vec{a}$ for some t. Solve your equation from (a) to find t.

$$\vec{a}^T(\vec{b} - \vec{p}) = 0$$
$$\vec{a}^T(\vec{b} - t\vec{a}) = 0$$
$$\vec{a}^T\vec{b} - t\,\vec{a}^T\vec{a} = 0$$
$$\vec{a}^T\vec{b} = t\,\vec{a}^T\vec{a} \implies t = \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}$$

(c) Put everything together to tell me the formula for $\vec{p}$ in terms of $\vec{a}$ and $\vec{b}$.

$$\vec{p} = t\vec{a} = \left(\frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}\right)\vec{a}.$$

Problem 2. Consider the following three data points:

$$\begin{pmatrix} t \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

(a) Write down the silly equation $A\vec{x} = \vec{b}$ which says that all three points are on the same line $C + tD = b$. (The silly equation has no solution.)

$$\begin{cases} C + (-1)D = 0 \\ C + 0D = 0 \\ C + 1D = 3 \end{cases} \Longrightarrow \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}.$$

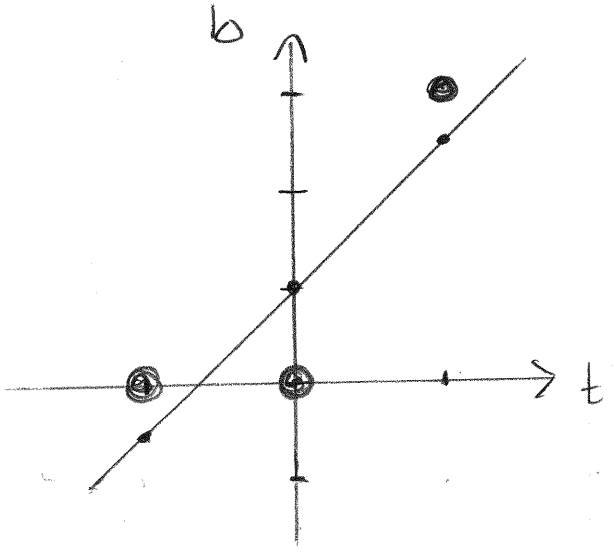
(b) Solve the associated "normal equation" $A^T A\vec{x} = A^T\vec{b}$.

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ 3/2 \end{pmatrix}.$$

(c) Draw the three data points together with the best fit line $C + tD = b$.



The best fit line is

$$b = 1 + \frac{3}{2}t.$$