Homework 5

4.1.24: Suppose A^{-1} exists. Let $\vec{a}_1^T, \dots, \vec{a}_n^T$ be the rows of A and let $\vec{b}_1, \dots, \vec{b}_n$ be the columns of A^{-1}. We have

$$AA^{-1} = I.$$

By looking at the (i,j) entry on each side we get

$$(\text{ith row } A)(\text{jth col } A^{-1}) = (i,j \text{ entry } I)$$

$$\vec{a}_i \circ \vec{b}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

In particular we see that $\vec{a}_i \circ \vec{b}_1 = 0$ for $i = 2, 3, \dots, n$. This means that the 1st column of A^{-1} (i.e. $\vec{b}_1$) is perpendicular to the 2nd, 3rd, $\dots$, nth rows of A.

4.1.25. Suppose A has columns

$$\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$$

such that $\vec{a}_i \circ \vec{a}_j = 0$ for $i \neq j$ and $\vec{a}_i \circ \vec{a}_i = \|\vec{a}_i\|^2 = 1$ for all i.

The ith row of A^T is $\vec{a}_i^{\,T}$ by definition
Then the i,j entry of A^TA is

$$
\begin{aligned}
(i,j)\text{-entry } A^TA &= (i\text{th row } A^T)(j\text{th col } A) \\
&= \vec{a}_i^{\,T}\vec{a}_j \\
&= \vec{a}_i \circ \vec{a}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}.
\end{aligned}
$$

In other words, $A^TA = I$.

4.1.26. For example, consider the matrix

$$
A = \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}.
$$

One can check that the columns of A are mutually perpendicular. It follows from 2.4.25 that A^TA is a "diagonal" matrix. In fact we have

$$
A^TA = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = 9I.
$$

4.2.1. Project $\vec{b}$ onto the line $t\vec{a}$:

(a) $\vec{b} = (1,2,2)$ & $\vec{a} = (1,1,1)$.

$$\vec{p} = \left(\frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}\right)\vec{a} = \frac{(1\ 1\ 1)\begin{pmatrix}1\\2\\2\end{pmatrix}}{(1\ 1\ 1)\begin{pmatrix}1\\1\\1\end{pmatrix}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \frac{5}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

(b) $\vec{b} = (1,3,1)$ & $\vec{a} = (-1,-3,-1)$.

$$\vec{p} = \left(\frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}\right)\vec{a} = \frac{(-1\ -3\ -1)\begin{pmatrix}1\\3\\1\end{pmatrix}}{(1\ 3\ 1)\begin{pmatrix}1\\3\\1\end{pmatrix}}\begin{pmatrix}1\\3\\1\end{pmatrix}$$

$$= \frac{-11}{11}\begin{pmatrix}1\\3\\1\end{pmatrix} = \begin{pmatrix}-1\\-3\\-1\end{pmatrix}.$$

We got $\vec{p} = \vec{b}$. Why did that happen?

Answer: Because the point $\vec{b}$ was <u>already</u> on the line $t\vec{a}$.

Projecting twice does nothing.

4.2.5. $\vec{a}_1 = (-1, 2, 2)$ & $\vec{a}_2 = (2, 2, -1)$.

Projection matrices:

$$P_1 = \vec{a}_1 \vec{a}_1^T / \vec{a}_1^T \vec{a}_1 = \frac{\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} (-1 \ 2 \ 2)}{(-1 \ 2 \ 2) \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}}$$

$$= \frac{1}{9} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}$$

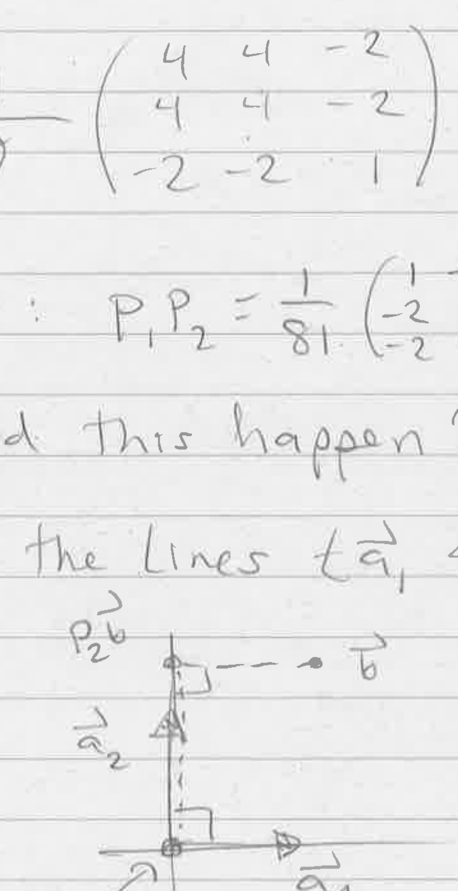
$$P_2 = \vec{a}_2^T \vec{a}_2 / \vec{a}_2^T \vec{a}_2 = \frac{\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} (2 \ 2 \ -1)}{(2 \ 2 \ -1) \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}}$$

$$= \frac{1}{9} \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}.$$

Product: $P_1 P_2 = \frac{1}{81} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix} \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Why did this happen?

Because the lines $t\vec{a}_1$ & $t\vec{a}_2$ are $\perp$:



We also have $P_2 P_1 = 0$.

4.2.10. We usually don't have $P_1 P_2 = P_2 P_1$.

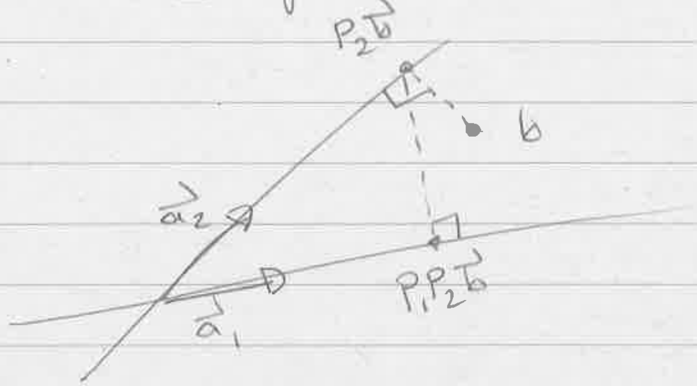
Example: $\vec{a}_1 = (1,0)$ & $\vec{a}_2 = (1,2)$.

$$P_1 = \frac{\begin{pmatrix}1\\0\end{pmatrix}(1\ 0)}{(1\ 0)\begin{pmatrix}1\\0\end{pmatrix}} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

$$P_2 = \frac{\begin{pmatrix}1\\2\end{pmatrix}(1\ 2)}{(1\ 2)\begin{pmatrix}1\\2\end{pmatrix}} = \frac{1}{5}\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}.$$

Product: $P_1 P_2 = \frac{1}{5}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix} = \frac{1}{5}\begin{pmatrix}1 & 2\\ 0 & 0\end{pmatrix}$.

This matrix first projects onto $t\vec{a}_2$ and then projects onto $t\vec{a}_1$:



Is $P_1 P_2$ a projection? I would say NO because

$$(P_1P_2)^2 = \frac{1}{25}\begin{pmatrix}1 & 2\\ 0 & 0\end{pmatrix} = \frac{1}{5}P_1P_2 \neq P_1P_2.$$

[It's a projection followed by a "contraction" by 1/5.]

4.2.16. Which linear combination of $(1,2,-1)$ & $(1,0,1)$ is closest to $(2,1,1)$?

Project $\vec{b} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ onto col space of $A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{pmatrix}$:

$$\text{proj} = A\vec{x} \quad \text{where}$$

$$A^T A \vec{x} = A^T \vec{b}$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} \vec{x} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} 1/6 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix}.$$

Then $\text{proj} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$

We projected $(2,1,1)$ and nothing happened because it was already in the plane!

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \frac{3}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

4.2.17 : Suppose that $P^2 = P$. Then we have

$$\begin{aligned}(I-P)^2 &= (I-P)(I-P) \\ &= I^2 - PI - IP + P^2 \\ &= I - P - P + P \\ &= I - P.\end{aligned}$$

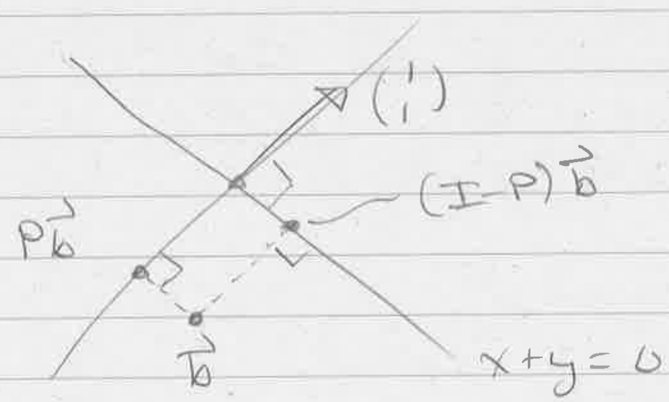
If P projects onto the column space of A then $I-P$ projects onto the space that is orthogonal to the column space of A.

[Remark: This is the space of vectors $\vec{x}$ that satisfy $A^T\vec{x} = \vec{0}$.]

4.2.18. Examples: If P projects onto the line $t(1,1)$

$$\left[\text{i.e. if } P = \frac{\binom{1}{1}(1\ 1)}{(1\ 1)\binom{1}{1}} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\right]$$

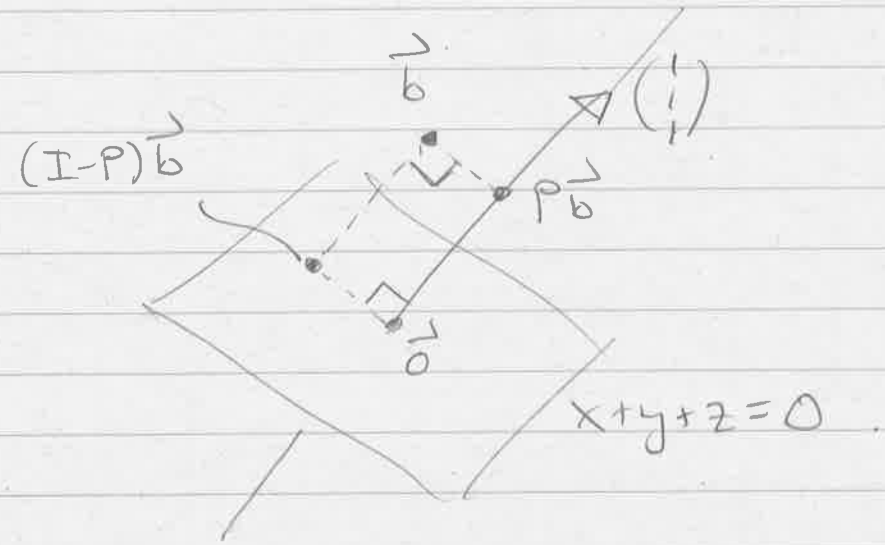
then $I-P$ projects onto the perpendicular line $x+y=0$.



If P projects onto the line $t(1,1,1)$

$\left[\text{i.e. if } P = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\right]$

then $I-P$ projects onto the <u>plane</u> $x+y+z=0$



4.2.19. To find two vectors in the plane $x-y-2z=0$, let y & z be <u>free</u>. Then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y+2z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

By setting $y=1, z=0$ & $y=0, z=1$ we find that

$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ are two vectors in the plane.

So we define $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Then the matrix that projects onto the plane is

$$P = A(A^TA)^{-1}A^T$$

$$= \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \left[\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \right]^{-1} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{6} \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 5 & -2 \\ 2 & -2 & 2 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{pmatrix}.$$

Is this correct? The next problem will provide a check.

4.2.20. The plane $x - y - 2z = 0$ has perpendicular line $t(1,-1,-2)$. The projection onto this line is

$$Q = \frac{\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} (1\ -1\ -2)}{(1\ -1\ -2)\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}} = \frac{1}{6}\begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{pmatrix}.$$

Therefore the projection onto the <u>plane</u> is

$$I - Q = \frac{1}{6}\begin{pmatrix} 6 & & \\ & 6 & \\ & & 6 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{pmatrix} \checkmark$$

4.3.5. We have 4 data points

$$\begin{pmatrix} t \\ b \end{pmatrix} = \begin{pmatrix} t_1 \\ 0 \end{pmatrix}, \begin{pmatrix} t_2 \\ 8 \end{pmatrix}, \begin{pmatrix} t_3 \\ 8 \end{pmatrix}, \begin{pmatrix} t_4 \\ 20 \end{pmatrix}$$

[Note: The times won't matter so Strang didn't even tell us what they are.]

Fit these points to a horizontal line of the form $C = b$.

The equations

$$\begin{aligned} C &= 0 \\ C &= 8 \\ C &= 8 \\ C &= 20 \end{aligned} \quad \leftrightarrow \quad \underset{A}{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}} \underset{\vec{x}}{\begin{pmatrix} C \end{pmatrix}} = \underset{\vec{b}}{\begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}}$$

obviously have no solution, so we try the normal equation $A^TA\hat{x} = A^T\vec{b}$:

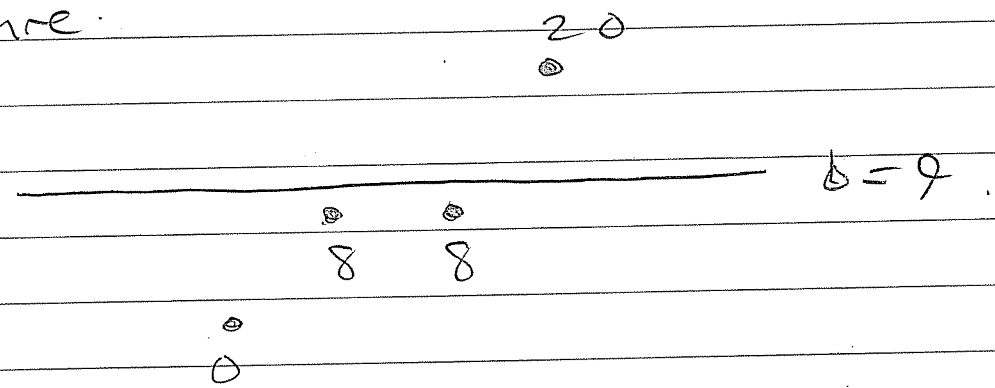
$$\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} C \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \\ 8 \\ 12 \end{pmatrix}$$

$$4C = 0 + 8 + 8 + 20$$

$$C = \frac{(0 + 8 + 8 + 20)}{4} = 9.$$

This $C = 9$ is just the <u>average</u> of the b values!

Picture:



4.3.7. Find the line $Dt = b$ through the origin <u>closest</u> to

$$\begin{pmatrix} t \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 8 \end{pmatrix}, \begin{pmatrix} 4 \\ 20 \end{pmatrix}$$

[Now strang tells us the t values]

The silly equations are

$$\begin{aligned} 0D &= 0 \\ 1D &= 8 \\ 3D &= 8 \\ 4D &= 20 \end{aligned} \longrightarrow \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix} (D) = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}$$

"$A\vec{x} = \vec{b}$"

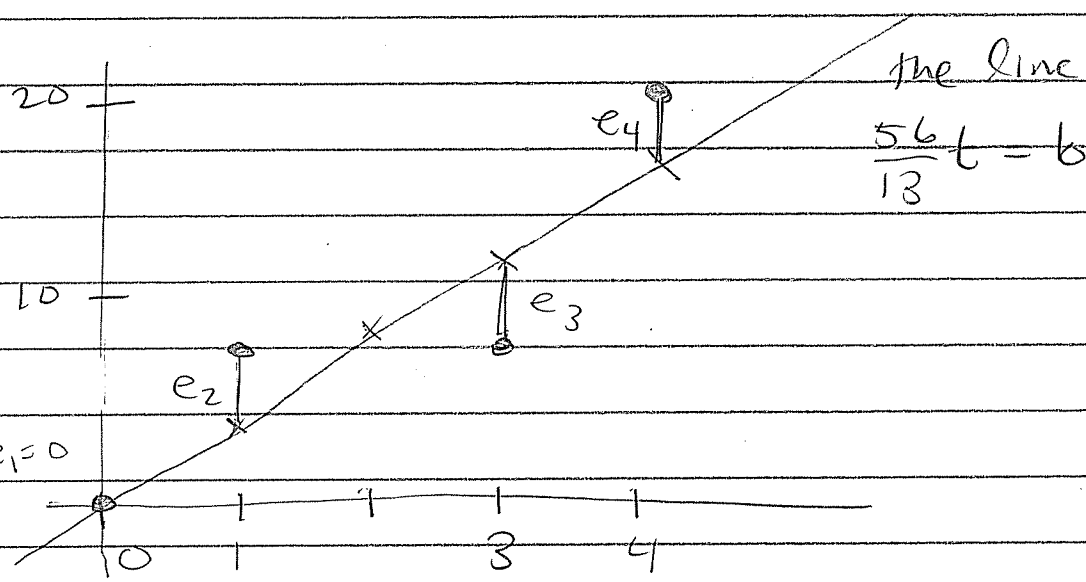
There is no solution, so we solve the normal equation $A^TA\hat{x} = A^T\vec{b}$

$$\begin{pmatrix}0&1&3&4\end{pmatrix}\begin{pmatrix}0\\1\\3\\4\end{pmatrix}(D) = \begin{pmatrix}0&1&3&4\end{pmatrix}\begin{pmatrix}0\\8\\8\\20\end{pmatrix}$$

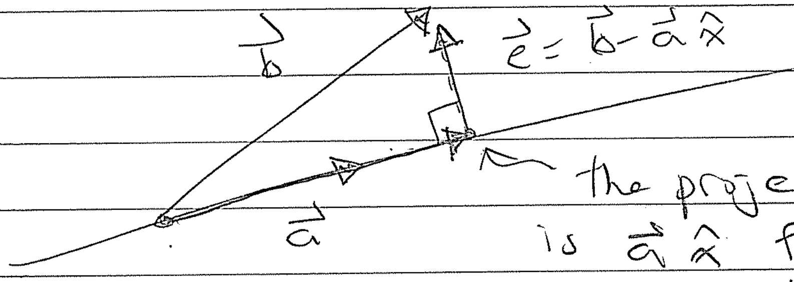
$$26D = 112$$

$$D = 112/26 = 56/13$$

Picture:



4.3.12. Project the data $\vec{b} = (b_1, \ldots, b_m)$ onto the line through $\vec{a} = (1, 1, \ldots, 1)$.



the projection is $\vec{a}\hat{x}$ for some <u>number</u> $\hat{x}$.

By orthogonality, we must have

$$\vec{a}^T \vec{e} = \vec{a}^T(\vec{b} - \vec{a}\hat{x}) = 0$$
$$\vec{a}^T\vec{b} - \vec{a}^T\vec{a}\,\hat{x} = 0$$
$$\vec{a}^T\vec{b} = \vec{a}^T\vec{a}\,\hat{x}.$$

(a) Solve this. $\vec{a}^T\vec{a}\,\hat{x} = \vec{a}^T\vec{b}$ is

$$(1\ 1 \cdots 1)\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}\hat{x} = (1\ 1 \cdots 1)\begin{pmatrix}b_1\\ \vdots\\ b_m\end{pmatrix}$$

$$m\hat{x} = b_1 + \cdots + b_m$$

$$\hat{x} = \frac{b_1 + \cdots + b_m}{m}$$

= the average/mean of the b values.

(b) The error vector is

$$\vec{e} = \vec{b} - \vec{a}\,\hat{x} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} - \hat{x}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} b_1 - \hat{x} \\ b_2 - \hat{x} \\ \vdots \\ b_m - \hat{x} \end{pmatrix}$$

Its squared length is the variance

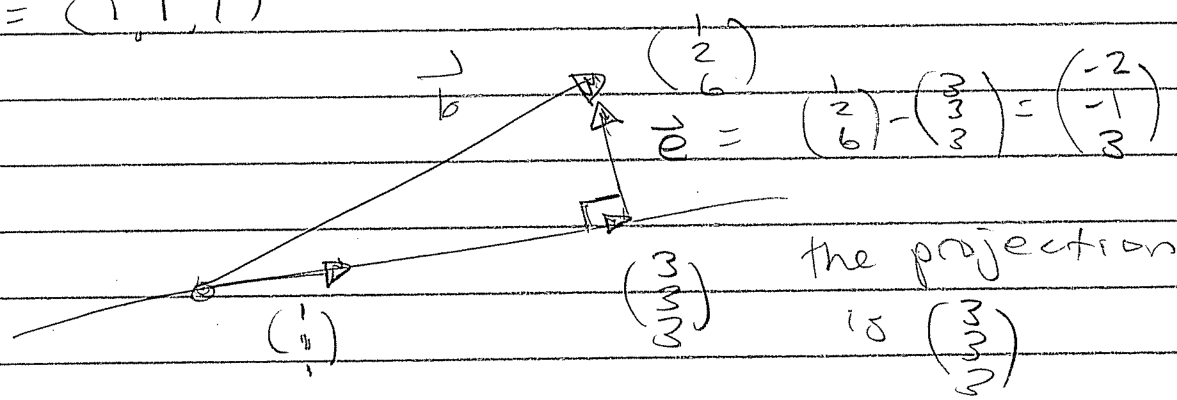
$$\|\vec{e}\|^2 = (b_1 - \hat{x})^2 + (b_2 - \hat{x})^2 + \cdots + (b_m - \hat{x})^2$$

Its length is the standard deviation

$$\|\vec{e}\| = \sqrt{(b_1 - \hat{x})^2 + \cdots + (b_m - \hat{x})^2}$$

(c) Example: $\vec{b} = (1, 2, 6)$

To find the best horizontal line (which is just the mean of the b's) we project $\vec{b}$ onto the line through $\vec{a} = (1, 1, 1)$



The error is $\perp$ to the projection

$$\begin{pmatrix} -2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = 0.$$

The matrix that peforms the projection onto $\vec{a} = (1, 1, 1)$ is

$$P = \frac{\vec{a}\,\vec{a}^T}{\vec{a}^T\vec{a}} = \frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}}{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Check:

$$P\vec{b} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

It works.

4.3.17. Find the line $C + Dt = b$ closest to the data points

$$\begin{pmatrix} t \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 21 \end{pmatrix}$$

Plug in:

$$\begin{aligned} C + D(-1) &= 7 \\ C + D(1) &= 7 \\ C + D(2) &= 21 \end{aligned} \rightarrow \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ 21 \end{pmatrix}$$

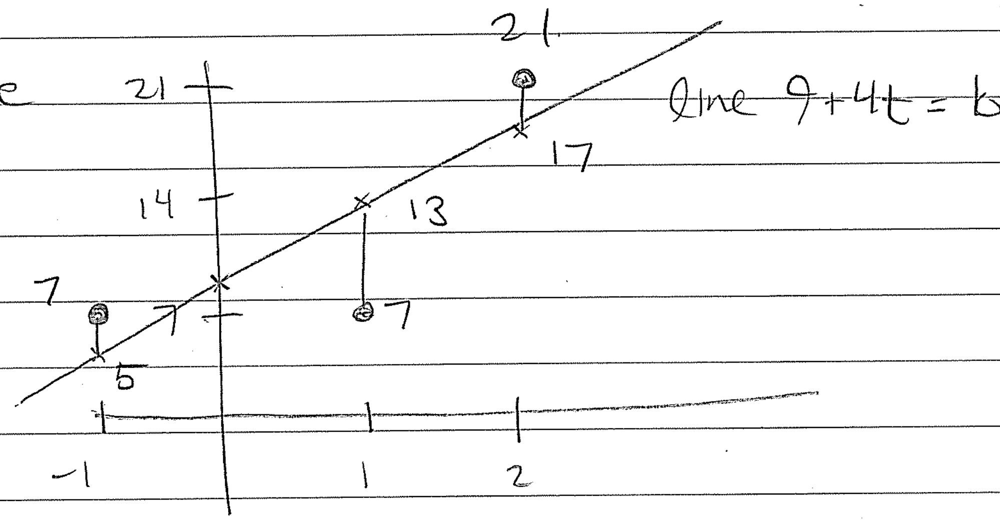
Least Squares:

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 7 \\ 7 \\ 21 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 35 \\ 42 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}.$$

Picture



4.3.22. Find the line $C+Dt=b$ closest to the points $\begin{pmatrix} t \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Normal Equation:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -2 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 5 \\ -10 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\boxed{\text{The line is } 1-t=b}$

Additional Problems (not assigned)

We say that P is a "projection matrix" if $P^T = P$ and $P^2 = P$.

A.1. Show $P = A(A^TA)^{-1}A^T$ is a projection matrix.

$P^2 = P$?

$$P^2 = [A(A^TA)^{-1}A^T][A(A^TA)^{-1}A^T]$$

$$= A(A^TA)^{-1}\cancel{(A^TA)(A^TA)^{-1}}A^T$$

$$= A(A^TA)^{-1}\, I \, A^T$$

$$= A(A^TA)^{-1}A^T = P \quad \checkmark$$

$P^T = P$?

$$P^T = (A(A^TA)^{-1}A^T)^T$$

$$= (A^T)^T[(A^TA)^{-1}]^T(A)^T$$

$$= A[(A^TA)^T]^{-1}A^T$$

$$= A[(A)^T(A^T)^T]^{-1}A^T$$

$$= A(A^TA)^{-1}A^T = P \quad \checkmark$$

A.2. If A is square and invertible, then

$$(A^T A)^{-1} = (A)^{-1}(A^T)^{-1}$$
$$= A^{-1}(A^T)^{-1}$$

Hence

$$P = A(A^T A)^{-1} A^T$$
$$= \cancel{A A^{-1}} \cancel{(A^T)^{-1} A^T}$$
$$= I \cdot I = I.$$

In this case we are projecting onto the full space. How do we project onto the full space? Do nothing! i.e. the identity function

A.3. If $P^T = P$ and $P^2 = P$, then

$$(I-P)^T = I^T - P^T = I - P \quad \checkmark$$ and

$$(I-P)^2 = (I-P)(I-P)$$
$$= II - PI - IP + PP$$
$$= I - P - P + P^2$$
$$= I - P \cancel{- P + P}$$
$$= I - P \quad \checkmark$$

Hence $I-P$ is also a projection.

A.4. The projections P and $I-P$ satisfy

$$P(I-P) = PI - P^2$$
$$= P - P = 0 \quad \left(\text{the matrix of all zeros}\right).$$

We say that the projection functions P and $I-P$ are "orthogonal" to each other. What does this mean?

Example: