HW 3 Solutions.

2.1.5. Consider the system

$$\begin{cases} x + y + z = 2 & (1) \\ x + 2y + z = 3 & (2) \\ 2x + 3y + 2z = 5. & (3) \end{cases}$$

The planes (1) & (2) meet along a line L. The plane (3) contains L because if x, y, z satisfy (1) & (2) they also satisfy (3).

Reason: (1) + (2) = (3).

To find the line we perform Gaussian elimination:

$$\begin{cases} x + y + z = 2 & (1) \to (1) \\ 0 + y + 0 = 1 & (2) \to (2) - (1) \end{cases}$$

$$\begin{cases} x + 0 + z = 1 & (1) \to (1) - (2) \\ 0 + y + 0 = 1 & (2) \to (2). \end{cases}$$

The pivots are x & y. Let z = t be free.

Then the solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-t \\ 1 \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

This is the line L.

2.1.6 If we replace ③ by

$$2x + 3y + 2z = 7 \qquad ③'$$

then this ③' is parallel to the original plane 3, so ③' does not touch the line L.

We conclude that the system of ①, ②, ③' has no solution.

2.1.7. Observe that the system

$$\begin{cases} x + y + z = 2 \\ x + 2y + z = 3 \\ 2x + 3y + 2z = 5 \end{cases}.$$

is equivalent to the vector equation

$$x\begin{pmatrix}1\\1\\2\end{pmatrix}+y\begin{pmatrix}1\\2\\3\end{pmatrix}+z\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}2\\3\\5\end{pmatrix}.$$

We already know that the solution is

$$(x,y,z)=(1-t,1,t) \text{ for any } t.$$

To solve

$$x\begin{pmatrix}1\\1\\2\end{pmatrix}+y\begin{pmatrix}1\\2\\3\end{pmatrix}+z\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}4\\6\\c\end{pmatrix}$$

compute the RREF:

$$\left(\begin{array}{ccc|c}1&1&1&4\\1&2&1&6\\2&3&2&c\end{array}\right)\Longrightarrow\left(\begin{array}{ccc|c}1&1&1&4\\0&1&0&2\\0&1&0&c-8\end{array}\right)$$

$$\Longrightarrow\left(\begin{array}{ccc|c}1&1&1&4\\0&1&0&2\\0&0&0&c-10\end{array}\right)$$

There is a solution only if $c=10$

2.1.8. 4 hyperplanes in 4D usually meet at a point.

4 vectors in 4D usually combine to produce any given vector $\vec{b}$.

Example:

$$x\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+y\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+z\begin{pmatrix}1\\1\\1\\0\end{pmatrix}+t\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}3\\3\\3\\2\end{pmatrix}$$

has the unique solution $(x,y,z,t)=(0,0,1,2)$.

2.2.5. $\begin{cases} 3x+2y=10 \\ 6x+4y=c \end{cases}$.

These are parallel lines so the system has either 0 or ∞ solutions.

RREF $\left(\begin{array}{cc|c} 3 & 2 & 10 \\ 6 & 4 & c \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 3 & 2 & 10 \\ 0 & 0 & c-20 \end{array}\right)$

No solutions when $c \neq 20$

∞ solutions when $c=20$.

2.2.7 $\begin{cases} ax + 3y = -3 \\ 4x + 6y = 6. \end{cases}$

If $a = 0$ then

$$\left(\begin{array}{cc|c} 0 & 3 & -3 \\ 4 & 6 & 6 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 4 & 6 & 6 \\ 0 & 3 & -3 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 4 & 0 & 12 \\ 0 & 3 & -3 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right) \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

If $a \neq 0$ then

$$\left(\begin{array}{cc|c} a & 3 & -3 \\ 4 & 6 & 6 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 3/a & -3/a \\ 4 & 6 & 6 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cc|c} 1 & 3/a & -3/a \\ 0 & 6-4\left(\frac{3}{a}\right) & 6-4\left(\frac{-3}{a}\right) \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 3/a & -3/a \\ 0 & (6a-12)/a & (6a+12)/a \end{array}\right).$$

If $a = 2$ then NO SOLUTION.

If $a \neq 2$ then unique solution.

2.2.11 On old HW3 solutions.

2.2.12

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 4 & 7 & 5 & 20 \\ 0 & -2 & 2 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 0 & 1 & 3 & 4 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 8 & 8 \end{array}\right] \qquad \begin{aligned} 2x + 3y + z &= 8 \\ y + 3z &= 4 \\ 8z &= 8 \end{aligned}$$

$$\Longrightarrow z = 1 \Longrightarrow y + 3(1) = 4$$
$$y = 1$$

$$\Longrightarrow 2x + 3(1) + 1(1) = 8$$
$$2x = 4$$
$$x = 2.$$

2.2.13

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 4 & -5 & 1 & 7 \\ 2 & -1 & -3 & 5 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & -3 & 2 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Longrightarrow z = 0 \Longrightarrow y + 0 = 1 \Longrightarrow 2x - 3 + 0 = 3$$
$$y = 1 \qquad 2x = 6$$
$$x = 3.$$

2.2.15 $\left[\begin{array}{ccc|c} 1 & b & 0 & 0 \\ 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & b & 0 & 0 \\ 0 & -2-b & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$

If $b=-2$ then we get

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

unique solution!

If $b \neq -2$ then we get

$$\left[\begin{array}{ccc|c} 1 & b & 0 & 0 \\ 0 & 1 & 1/(2+b) & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & b & 0 & 0 \\ 0 & 1 & 1/(2+b) & 0 \\ 0 & 0 & 1-\frac{1}{2+b} & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & b & 0 & 0 \\ 0 & 1 & 1/(2+b) & 0 \\ 0 & 0 & (1+b)/(2+b) & 0 \end{array}\right]$$

No solution if $b \neq -1$. So let $b=-1$. Then...

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$z=t$, $y=-t$, $x=-t$ for any t.