Math 210 **Summer 2017**
Homework 1 Solutions **Drew Armstrong**

Problems from 4th edition of Gilbert Strang's *Introduction to Linear Algebra.*

Section 1.1 solutions are included in the Week 1 Notes.

For the Section 1.2 problems, we recall the rules of vector arithmetic: for all vectors $\vec{u}, \vec{v}, \vec{w}$ and scalars a, b we have

$$\begin{aligned}
&\vec{u} + \vec{v} = \vec{v} + \vec{u}\\
&\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}\\
&a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}\\
&(a + b)\vec{u} = a\vec{u} + b\vec{u}\\
&\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}\\
&a(\vec{u} \bullet \vec{v}) = (a\vec{u}) \bullet \vec{v} = \vec{u} \bullet (a\vec{v})\\
&\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}.
\end{aligned}$$

More succinctly, the dot product distributes over arbitrary *linear combinations*:

$$\vec{u} \bullet (a\vec{v} + b\vec{w}) = a(\vec{u} \bullet \vec{v}) + b(\vec{u} \bullet \vec{w}).$$

1.2.4. For this problem we assume that $\vec{v}$ and $\vec{w}$ are *unit vectors*, i.e., that $\|v\| = \|\vec{w}\| = 1$. This implies that

$$\vec{v} \bullet \vec{v} = \|\vec{v}\|^2 = 1 \quad \text{and} \quad \vec{w} \bullet \vec{w} = \|\vec{w}\|^2 = 1.$$

Then we use the rules of vector arithmetic to compute the desired dot products.

(a)

$$\vec{v} \bullet (-\vec{v}) = \vec{v} \bullet (-1\vec{v}) = -1(\vec{v} \bullet \vec{v}) = -1(1) = -1.$$

(b)

$$\begin{aligned}
(\vec{v} + \vec{w}) \bullet (\vec{v} - \vec{w}) &= \vec{v} \bullet \vec{v} + \vec{w} \bullet \vec{v} + \vec{v} \bullet (-\vec{w}) + \vec{w} \bullet (-\vec{w})\\
&= \vec{v} \bullet \vec{v} + \vec{w} \bullet \vec{v} - \vec{v} \bullet \vec{w} - \vec{w} \bullet \vec{w}\\
&= \vec{v} \bullet \vec{v} + \vec{v} \bullet \vec{w} - \vec{v} \bullet \vec{w} - \vec{w} \bullet \vec{w}\\
&= \vec{v} \bullet \vec{v} - \vec{w} \bullet \vec{w}\\
&= 1 - 1\\
&= 0.
\end{aligned}$$

(c)

$$\begin{aligned}
(\vec{v} + 2\vec{w}) \bullet (\vec{v} - 2\vec{w}) &= \vec{v} \bullet \vec{v} + (2\vec{w}) \bullet \vec{v} + \vec{v} \bullet (-2\vec{w}) + (2\vec{w}) \bullet (-2\vec{w})\\
&= \vec{v} \bullet \vec{v} + 2(\vec{w} \bullet \vec{v}) - 2(\vec{v} \bullet \vec{w}) - 4(\vec{w} \bullet \vec{w})\\
&= \vec{v} \bullet \vec{v} + 2(\vec{v} \bullet \vec{w}) - 2(\vec{v} \bullet \vec{w}) - 4(\vec{w} \bullet \vec{w})\\
&= \vec{v} \bullet \vec{v} - 4(\vec{w} \bullet \vec{w})\\
&= 1 - 4\\
&= -3.
\end{aligned}$$

1.2.5. Consider the vectors $\vec{v} = (3,1)$ and $\vec{w} = (2,1,2)$. We can find unit vectors in the same directions as follows:

$$\vec{u}_1 = \frac{\pm 1}{\|\vec{v}\|}\vec{v} = \frac{\pm 1}{\sqrt{10}}\begin{pmatrix}3\\1\end{pmatrix} \quad \text{and} \quad \vec{u}_2 = \frac{\pm 1}{\|\vec{w}\|}\vec{w} = \frac{\pm 1}{\sqrt{9}}\begin{pmatrix}2\\1\\2\end{pmatrix}$$

To find vectors that are **perpendicular** to $\vec{u}_1$ and $\vec{u}_2$ we first observe that $(-1,3)$ is perpendicular to $\vec{v} = (3,1)$ (and also to $\vec{u}_1$) because their dot product is zero:

$$\begin{pmatrix}-1\\3\end{pmatrix} \bullet \begin{pmatrix}3\\1\end{pmatrix} = (-1)3 + 3(1) = 0.$$

Then to find a **unit** vector perpendicular to $\vec{u}_1$ we take

$$\vec{U}_1 = \frac{\pm 1}{\left\|\begin{pmatrix}-1\\3\end{pmatrix}\right\|}\begin{pmatrix}-1\\3\end{pmatrix} = \frac{\pm 1}{\sqrt{10}}\begin{pmatrix}-1\\3\end{pmatrix}.$$

To find a vector perpendicular to $\vec{w} = (2,1,2)$ consider any $\vec{x} = (x,y,z)$ and suppose that we have

$$\begin{pmatrix}2\\1\\2\end{pmatrix} \bullet \begin{pmatrix}x\\y\\z\end{pmatrix} = 0$$
$$2x + y + 2z = 0.$$

There is a whole plane of possible solutions to we just pick one at random: say $\vec{x} = (1,0,-1)$. Then we define

$$\vec{U}_2 = \frac{\pm 1}{\left\|\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\|}\begin{pmatrix}1\\0\\-1\end{pmatrix} = \frac{\pm 1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}.$$

1.2.6.

(a) Describe every vector $\vec{w} = (w_1, w_2)$ that is perpendicular to $\vec{v} = (2,-1)$.
They form a line. The equation of the line is $2w_1 - w_2 = 0$.

(b) The vectors that are pendicular to $\vec{V} = (1,1,1)$ lie on a
plane.

(c) The vectors that are perpendicular to $(1,1,1)$ and $(1,2,3)$ lie on a
line.
Reason: The vectors perpendicular to $(1,1,1)$ form a plane and the vectors perpendicular to $(1,2,3)$ form a different plane. The vectors perpendicular to both form the intersection of these two planes, which is a line. In class we showed that this line can be described as $t(1,-2,1)$, where the vector $(1,-2,1)$ is called the *cross product* of $(1,1,1)$ and $(1,2,3)$.

1.2.7. Find the angle θ between each pair of vectors $\vec{v}, \vec{w}$. Recall that the angle satisfies the equation

$$\cos\theta = \frac{\vec{v} \bullet \vec{w}}{\|v\| \cdot \|\vec{w}\|} = \frac{\vec{v} \bullet \vec{w}}{\sqrt{\vec{v} \bullet \vec{v}} \cdot \sqrt{\vec{w} \bullet \vec{w}}}$$

(a) For $\vec{v} = (1, \sqrt{3})$ and $\vec{w} = (1, 0)$ we have $\vec{v} \bullet \vec{v} = 4$, $\vec{w} \bullet \vec{w} = 1$ and $\vec{v} \bullet \vec{w} = 1$. Hence we get

$$\cos\theta = \frac{1}{\sqrt{4} \cdot \sqrt{1}} = \frac{1}{2} \quad \Longrightarrow \quad \theta = \pm 60^\circ$$

(b) For $\vec{v} = (2, 2, -1)$ and $\vec{w} = (2, -1, 2)$ we have $\vec{v} \bullet \vec{v} = 9$, $\vec{w} \bullet \vec{w} = 9$ and $\vec{v} \bullet \vec{w} = 0$. Hence we get

$$\cos\theta = \frac{0}{\sqrt{9} \cdot \sqrt{9}} = 0 \quad \Longrightarrow \quad \theta = \pm 90^\circ$$

(c) For $\vec{v} = (1, \sqrt{3})$ and $\vec{w} = (-1, \sqrt{3})$ we have $\vec{v} \bullet \vec{v} = 4$, $\vec{w} \bullet \vec{w} = 4$ and $\vec{v} \bullet \vec{w} = 2$. Hence we get

$$\cos\theta = \frac{2}{\sqrt{4} \cdot \sqrt{4}} = \frac{1}{2} \quad \Longrightarrow \quad \theta = \pm 60^\circ$$

(d) For $\vec{v} = (3, 1)$ and $\vec{w} = (-1, -2)$ we have $\vec{v} \bullet \vec{v} = 10$, $\vec{w} \bullet \vec{w} = 5$ and $\vec{v} \bullet \vec{w} = -5$. Hence we get

$$\cos\theta = \frac{-5}{\sqrt{10} \cdot \sqrt{5}} = \frac{-1}{\sqrt{2}} \quad \Longrightarrow \quad \theta = \pm 135^\circ$$

1.2.8. True or False?

(a) If $\vec{u}$ is perpendicular (in three dimensions) to $\vec{v}$ and $\vec{w}$, those vectors $\vec{v}$ and $\vec{w}$ are parallel. Algebraically we assume that $\vec{u} \bullet \vec{v} = 0$ and $\vec{u} \bullet \vec{w} = 0$ and the statement is that there exists some scalar t such that $\vec{v} = t\vec{w}$.

FALSE. For example, consider the standard basis vectors $\vec{u} = (1, 0, 0)$, $\vec{v} = (0, 1, 0)$ and $\vec{w} = (0, 0, 1)$. Then $\vec{u}$ is perpendicular to both $\vec{v}$ and $\vec{w}$ because

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \bullet \begin{pmatrix}0\\1\\0\end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix}1\\0\\0\end{pmatrix} \bullet \begin{pmatrix}0\\0\\1\end{pmatrix} = 0$$

However, $\vec{v}$ and $\vec{w}$ are not parallel because there is no scalar t for which

$$\begin{pmatrix}0\\1\\0\end{pmatrix} = t\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\t\end{pmatrix}.$$

The reason is because $1 \neq 0$. [Remark: The same statement is TRUE for vectors in the plane. Can you prove it?]

(b) If $\vec{u}$ is perpendicular to $\vec{v}$ and $\vec{w}$, then $\vec{u}$ is perpendicular to $\vec{v} + 2\vec{w}$. Algebraicaly we assume that $\vec{u} \bullet \vec{v} = 0$ and $\vec{u} \bullet \vec{w} = 0$ and the statement says that $\vec{u} \bullet (\vec{v} + 2\vec{w}) = 0$.

This is TRUE because

$$\vec{u} \bullet (\vec{v} + 2\vec{w}) = (\vec{u} \bullet \vec{v}) + 2(\vec{u} \bullet \vec{w}) = 0 + 2(0) = 0.$$

(c) If $\vec{u}$ and $\vec{v}$ are perpendicular unit vectors then $\|\vec{u} - \vec{v}\| = \sqrt{2}$.

TRUE. The fact that $\vec{u}$ and $\vec{v}$ are perpendicular unit vectors means that $\vec{u} \bullet \vec{u} = 1$, $\vec{u} \bullet \vec{v} = 0$ and $\vec{v} \bullet \vec{v} = 0$. Then we have

$$\begin{aligned}\|\vec{u} - \vec{v}\|^2 &= (\vec{u} - \vec{v}) \bullet (\vec{u} - \vec{v}) \\ &= \vec{u} \bullet \vec{u} - 2(\vec{u} \bullet \vec{v}) + \vec{v} \bullet \vec{v} \\ &= 1 - 2(0) + 1 \\ &= 2.\end{aligned}$$