Math 210 F **Spring 2016**

Homework 7 **Drew Armstrong**

Let A be a square matrix. Recall that an eigenvector of A is a vector $\vec{x} \neq \vec{0}$ such that $A\vec{x} = \lambda\vec{x}$ for some number λ. In this case we say that λ is the eigenvalue of $\vec{x}$.

Problem 1. Eigenvalues of Geometric Transformations.

(a) The rotation matrix R_θ **rarely** has any (real) eigenvalues. For which angles θ does it have real eigenvalues, and what are the eigenvalues in these cases?

(b) Let A be any matrix such that $(A^TA)^{-1}$ exists. Draw a picture to show that the only eigenvalues the projection matrix $P = A(A^TA)^{-1}A^T$ are 1 and 0.

(c) Let P be some matrix whose eigenvalues are 1 and 0 (maybe P is a projection matrix). In this case, show that the eigenvalues of the matrix $2P - I$ are 1 and -1. [Hint: Suppose that $P\vec{x} = 0\vec{x}$ and $P\vec{y} = 1\vec{y}$ for some (nonzero) eigenvectors $\vec{x}$ and $\vec{y}$. Then $(2P - I)\vec{x} =?$ and $(2P - I)\vec{y} =?$.]

Problem 2. Fibonacci Numbers. Consider any 2×2 matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We saw in class that the number λ is an eigenvalue of A precisely when the matrix

$$A - \lambda I_2 = \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}$$

is **not** invertible. [Reason: Because the equation $(A - \lambda I_2)\vec{x} = \vec{0}$ has a non-trivial solution $\vec{x} \neq \vec{0}$ only when the matrix $(A - \lambda I_2)$ has some non-trivial column relation.]

(a) Write down the formula for the inverse of the 2×2 matrix $A - \lambda I_2$ and use it to explain why the inverse fails to exist precisely when $(a - \lambda)(d - \lambda) - bc = 0$. This is called the characteristic equation of A.

(b) Now consider the "Fibonacci matrix" $T = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ from class. Solve the characteristic equation to show that the eigenvalues of T are

$$\varphi_1 := \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \varphi_2 := \frac{1 - \sqrt{5}}{2}.$$

(c) Solve the linear systems $T\vec{u} = \varphi_1\vec{u}$ and $T\vec{v} = \varphi_2\vec{v}$ to find the corresponding eigenvectors $\vec{u}$ and $\vec{v}$. [Hint: It will be helpful to use the identities $\varphi_1^2 = \varphi_1 + 1$ and $\varphi_2^2 = \varphi_2 + 1$.]

(d) Solve the linear system

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \vec{u} & \vec{v} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = a\vec{u} + b\vec{v}$$

to express the initial condition vector $\vec{f_0} = (f_1, f_0) = (1, 0)$ in terms of the two eigenvectors $\vec{u}$ and $\vec{v}$ from part (c). [In class this led us to a (surprising) formula for the n-th Fibonacci number: $f_n = [(1 + \sqrt{5})^n - (1 - \sqrt{5})^n]/(2^n\sqrt{5})$.]

(e) Finally, draw the line $t\vec{u}$ in the plane $\mathbb{R}^2$ along with the first several points $\vec{f_0}$, $\vec{f_1}$, $\vec{f_2}$, $\ldots$, $\vec{f_5}$. What do you see?