MTH 210, Spring 2016
HW4 Solutions.

Problem 1: Consider the following

$$A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & -2 & 0 \end{pmatrix} \quad \& \quad \vec{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

(a) Compute $A\vec{x}$ as a linear combination of the columns of $A$:

$$A\vec{x} = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$= 1 \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \end{pmatrix} + (-1) \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \\ -4 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 3 \\ -3 \end{pmatrix}.$$

(b) Compute $A\vec{x}$ by taking the dot product of $\vec{x}$ with the rows of $A$:

$$A\vec{x} = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} (2\ 0\ 1)\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \\ (-1\ 0\ 2)\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \\ (0\ 1\ -1)\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \\ (1\ -2\ 0)\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 2+0-1 \\ -1+0-2 \\ 0+2+1 \\ 1-4+0 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 3 \\ -3 \end{pmatrix} \checkmark$$

same answer :)

Problem 2: (a) Find the matrices $I, P, R$ such that for all $x$ & $y$ we have

$$I\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad P\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}, \quad R\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}.$$

$$I\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1x + 0y \\ 0x + 1y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Longrightarrow I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$P\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 0x + 1y \\ 1x + 0y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Longrightarrow P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$R\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} 0x - 1y \\ 1x + 0y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Longrightarrow R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(b) Find the matrix $A$ such that for all $x, y, z$ we have

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+y \\ x+2z \end{pmatrix}.$$

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+y \\ x+2z \end{pmatrix} = \begin{pmatrix} 2x+1y+0z \\ 1x+0y+2z \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Longrightarrow A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}. \quad ///$$

Problem 3: Consider the following

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

(a) Compute the vector $\vec{v} = B\vec{x}$. I'll use the column method to get

$$\vec{v} = B\vec{x} = \begin{pmatrix} 3 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3x + y \\ x \\ -y \end{pmatrix}.$$

(b) Compute $A\vec{v} = A(B\vec{x})$. I'll use the column method again to get

$$A\vec{v} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3x + y \\ x \\ -y \end{pmatrix}$$

$$= (3x+y)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + x\begin{pmatrix} 0 \\ -1 \end{pmatrix} + (-y)\begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3x+y \\ 3x+y \end{pmatrix} + \begin{pmatrix} 0 \\ -x \end{pmatrix} + \begin{pmatrix} -2y \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3x - y \\ 2x + y \end{pmatrix}.$$

(c) Find the matrix $C$ such that for all $\vec{x}$ we have $C\vec{x} = A(B\vec{x})$ :

$$C\begin{pmatrix} x \\ y \end{pmatrix} = A\left(B\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 3x - y \\ 2x + y \end{pmatrix}$$

$$\Longrightarrow \quad C = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}.$$

What would be a good name for this matrix? I think I will call it

$$C = \text{“}AB\text{”}.$$

Problem 3: (a) Let $A$ have shape $3\times 5$, $B$ have shape $5\times 3$ & $C$ have shape $3\times 2$. Then

- $AB$ is defined with shape $3\times 3$
  $\underbrace{3\times 5 \text{ — } 5\times 3}$
- $BA$ is defined with shape $5\times 5$
  $\underbrace{5\times 3 \text{ — } 3\times 5}$
- $ABC$ is defined with shape $3\times 2$
  $\underbrace{3\times 5 \text{ — } 5\times 3 \text{ — } 3\times 2}$

- $\underset{3\times 2}{C}\;\underset{5\times 3}{B}\;\underset{3\times 5}{A}$ is not defined because $2 = \#\text{cols of } C \neq \#\text{rows of } B = 5$

- $\underset{2\times 3}{C^T}\;\underset{5\times 3}{B}\;\underset{3\times 5}{A}$ is not defined because $3 = \#\text{cols of } C^T \neq \#\text{rows of } B = 5$.

(1b) Now let $A$ have shape $m\times n$ and let $B$ have shape $p\times q$.

- TRUE. If $A^2 = AA$ is defined then $n = \#\text{cols of } A = \#\text{rows of } A = m$, hence $A$ is square.

- FALSE. The matrix $A^TA$ is always defined with shape $n\times n$, even when $A$ is not square.

- TRUE. If $AB = B$ then $AB$ is defined so we must have $n = \#\text{cols } A = \#\text{rows } B = p$. Then $AB$ has shape $m\times q$. But since $AB = B$, $AB$ also has shape $p\times q$. We conclude that $m = p = n$, hence $A$ is square.

- FALSE. If $AB = B$ then we know from the previous that $A$ is square (i.e. $m=n$) and $n = \#\text{cols } A = \#\text{rows } B = p$. But there is no reason for $p=q$.

- FALSE. If $m=q$ & $n=p$ then $AB$ is defined (with shape $m\times q = m\times m = q\times q$) and $BA$ is defined (with shape $p\times n = n\times n = p\times p$), but there is no reason for $m=n$ or $p=q$.

- TRUE. If $AB$ & $BA$ are both defined then we saw in the previous that $AB$ & $BA$ are both square.

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Problem 5.

(a) Using our formula from class gives

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1+2 & 1+2 \\ 3+4 & 3+4 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 7 & 7 \end{pmatrix}$$

&

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1+3 & 2+4 \\ 1+3 & 2+4 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 4 & 6 \end{pmatrix}$$

which are NOT the same matrix.

(b) Consider the matrix equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} a+b & a+b \\ c+d & c+d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ a+c & b+d \end{pmatrix},$$

which is equivalent to a system of 4 linear equations in 4 unknowns:

$$\begin{cases} a+b = a+c \\ c+d = a+c \\ a+b = b+d \\ c+d = b+d \end{cases}$$

Luckily it's a rather easy system so we don't need the full power of Gaussian elimination. The general solution is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} s & t \\ t & s \end{pmatrix} = s\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with free parameters $s$ & $t$.

The Moral:

If $AB$ & $BA$ are both defined and have the same shape (i.e. if $A$ & $B$ are both square of the same shape) then we <u>might</u> have

$$AB = BA,$$

but probably not. This is the one big difference between the algebra of matrices and the algebra of numbers.

I prefer to view this as an advantage for matrix algebra because it allows us to represent more interesting kinds of things (such as compositions of rotations in 3D space, which is not commutative).