MTH 210 F
Intro. to Linear Algebra.

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All lecture notes and course information
will be posted on my webpage :

www. math. miami. edu / ~ armstrong

For extra reading and practice problems
I recommend Gilbert Strang's
"Intro. to Linear Algebra", 4th ed.,
but this text is not required.

I also have lecture notes from Spring 2013
posted on my webpage.

Your grade will be based on :

Homework	20%
Exam 1	25%
Exam 2	25%
Final Exam	80%

What is this course about?

Linear Algebra is the common denominator of all mathematics. From the most pure to the most applied, if you use mathematics then you will use Linear Algebra.

The importance of Calculus has plateaued but Linear Algebra continues to gain ground as computers and data become more important.

Before discussing applications of Linear Algebra we must first develop the languge, which is based on "matrices" and "vectors". These, in turn, are based on "coordinate geometry", so that's where we'll begin.

BEGIN.

The subject of geometry is fundamentally about points in space. But what is "space", and what is a "point"?
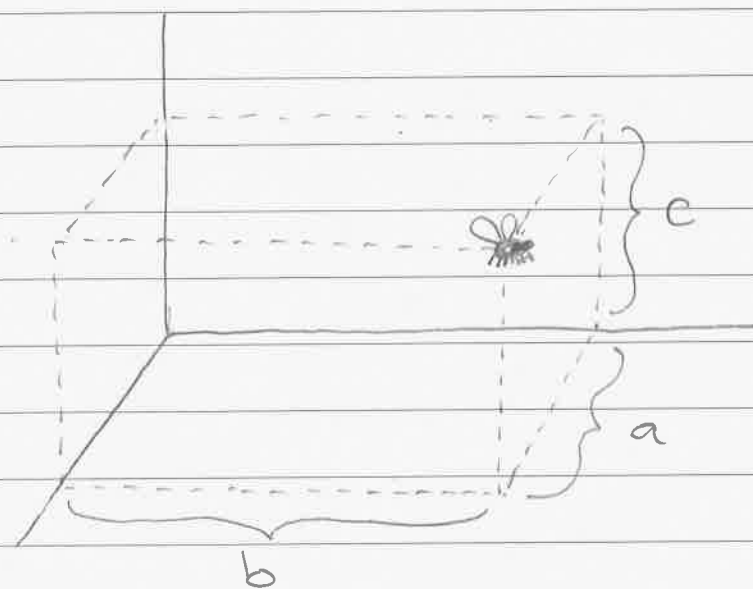
Our modern understanding is based on
a revolutionary idea of René Descartes
and Pierre de Fermat from the early 1600s.

⭐ Revolutionary Idea :

A point is an ordered list of numbers.

What ?!

Apparently Descartes was lying in bed
and he saw a fly buzzing in the corner.

He imagined that the fly was at the
corner of a rectangular box with
dimensions a, b, c.

Descartes realized that the numbers
a, b, c (in some fixed order) uniquely
specify the position of the fly!

$$(a, b, c) = \text{the "(Des) cartesian coordinates"}$$
$$\text{of the fly.}$$

In this class we will write the coordinates
as a vertical column

$$\vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

and we will call this a vector. By
also allowing negative numbers we can
uniquely specify the position of any
point in space. So we have

$$\text{point} = \text{vector} = \text{vertical column of}$$
$$\text{numbers .}$$

The point with all coordinates $= 0$ is
a very special point called the origin
of the coordinate system.

But this is more than just a notation because it suggests new kinds of things we can do to points.

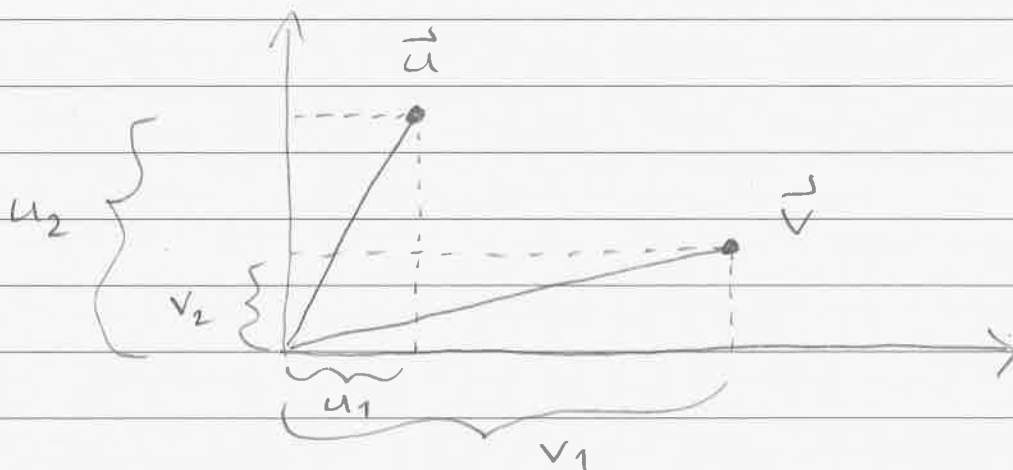For example, we can "add" them.

Given vectors $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ & $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$
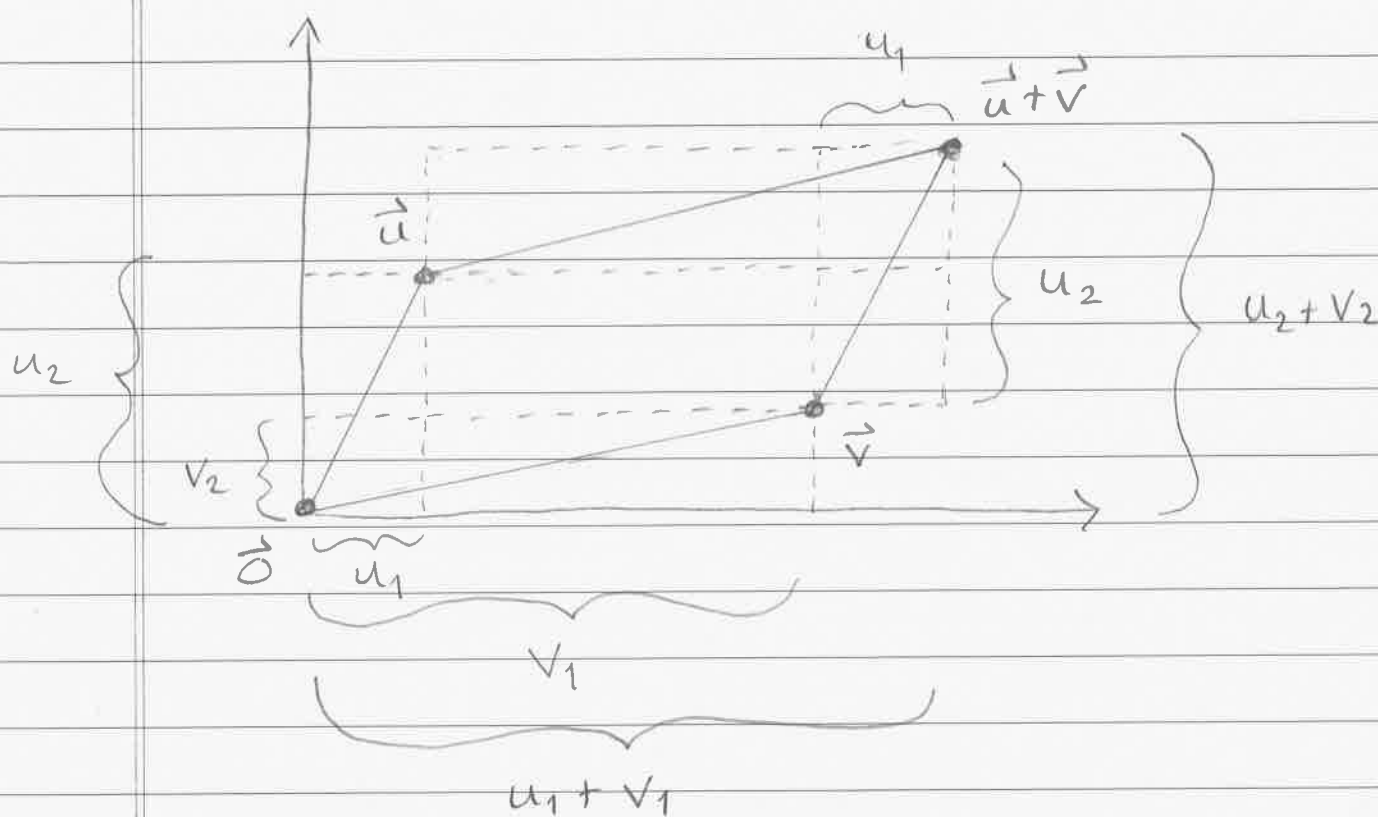
we will define

$$\vec{u} + \vec{v} \underset{\uparrow}{:=} \begin{pmatrix} u_1 + u_2 \\ v_1 + v_2 \end{pmatrix} .$$

definition

That is, we add two vectors by adding their respective components (or entries). This definition seems obvious in terms of numbers, but what does it mean geometrically? Let's first draw the points $\vec{u}$ & $\vec{v}$.

Then we have

Notice that the points $\vec{0} := \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \vec{u}, \vec{v}$ and $\vec{u} + \vec{v}$ form the four vertices of a parallelogram. This is called the
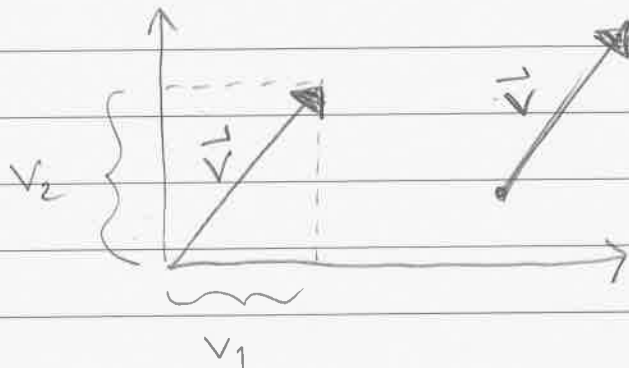
Parallelogram Law

of vector addition.

The parallelogram law suggests a
subtle but very useful idea

☆ Subtle Idea :

Sometimes we will think of the vector
$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ as an arrow with head at
the point $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and tail at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

The subtle thing is that we're allowed
to pick up the arrow and move it,
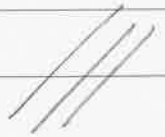as long as we don't change its
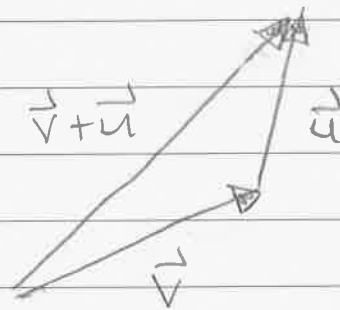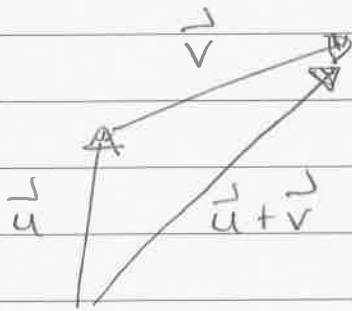length or direction

The useful thing is that this makes
addition and subtraction of vectors
very easy to describe.
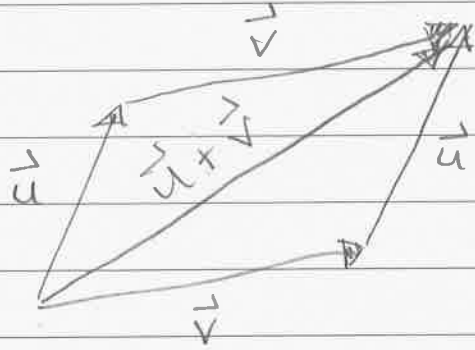
Vectors add "head-to-tail". Note from
the picture that addition of vectors
is commutative; that is, we have

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$

Question : How can we use this idea
to subtract vectors ?

I plan to have the webpage up by Friday at :

www.math.miami.edu/varmstrong/210sp16.html

[It's not there yet.] I will also assign. HW1 on Friday and it will be due next Fri Jan 22 in class. No late HW will be accepted.

Note that there is NO CLASS next Monday [MLK day].

Recall from last time :

A vector is a vertical column of numbers

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Here we say that n is the dimension of the vector $\vec{v}$. Vectors of the same dimension can be added as follows :

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} := \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{pmatrix}$$
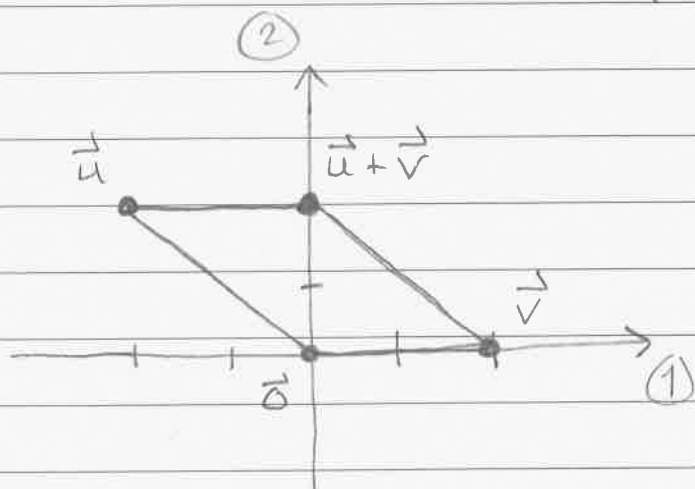
If $n=2$ or $n=3$ then vector addition has a nice geometric interpretation.

☆ Parallelogram Law: Let $\vec{u}$ & $\vec{v}$ be vectors in 2 or 3 dimensions, thought of as points in Cartesian coordinates. Then the four points

$$\vec{0}, \quad \vec{u}, \quad \vec{v}, \quad \vec{u} + \vec{v}$$

are the vertices of a parallelogram.

Example: Let $\vec{u} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ & $\vec{v} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.
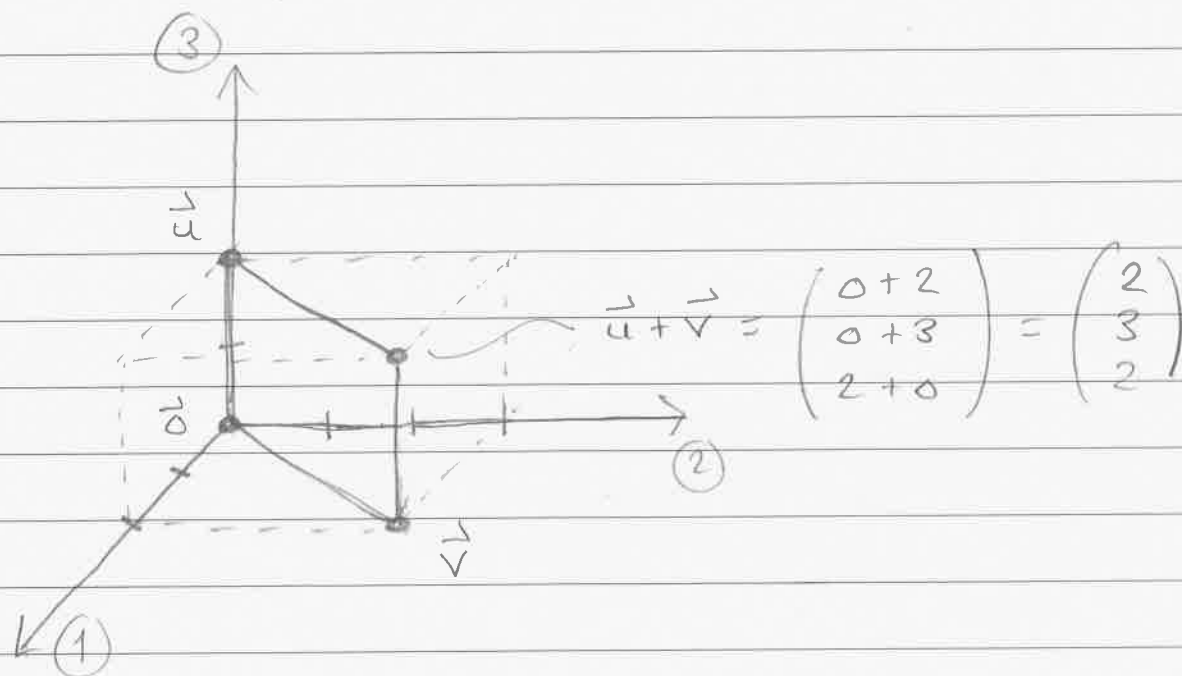

See the parallelogram?

Note that

$$\vec{u} + \vec{v} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 + 2 \\ 2 + 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

agrees with the picture.

Example : $\vec{u} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$ & $\vec{v} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$

$$\vec{u} + \vec{v} = \begin{pmatrix} 0 + 2 \\ 0 + 3 \\ 2 + 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

See the 2D parallelogram living in 3D ?
[It's actually a rectangle .]

The pictures for $n \geq 4$ are harder
to draw. "

There is another important way to think
about vector addition

☆ Subtle Idea : Sometimes we will think
of the vector

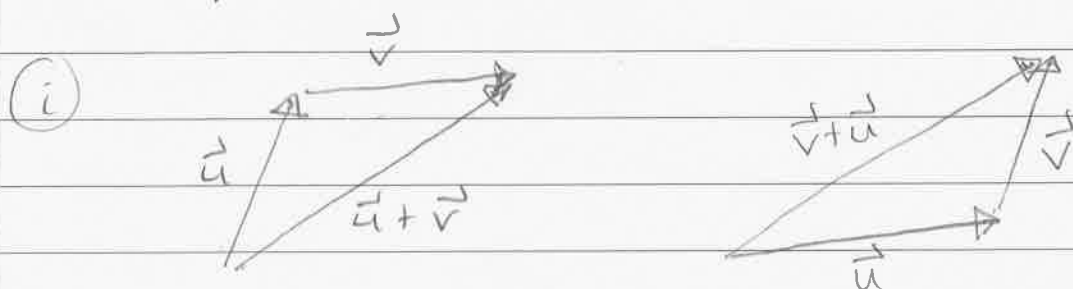$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

as an arrow with tail at $(0, 0, \cdots, 0)$
and head at $(v_1, v_2, \cdots, v_n)$.

[Remark : Sometimes I will write a
vector as a list with commas. This
is just to save space ; it's still a
vertical column in my mind.   ]

The subtle thing is that we can pick
up the arrow and move it around ,
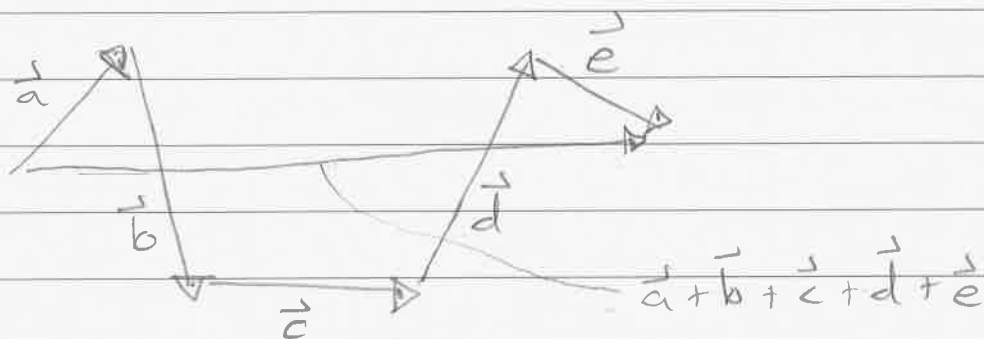as long as we don't change its
length or direction .

In this languge, we can add vectors
"head-to-tail"

Examples :

(i)

Note that $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ because the
are both the diagonal of the same
parallelogram.

(ii) It works with multiple vectors also:

$$\vec{a} + \vec{b} + \vec{c} + \vec{d} + \vec{e}$$

Adding the vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}$ in any
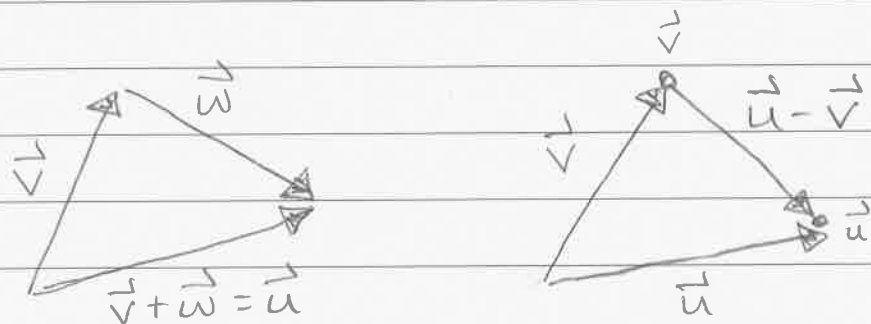order [there are 120 ways to do this]
gives the same resut!

This language also suggests how to
subtract vectors.

Consider two vectors $\vec{u}$ & $\vec{v}$ of the same dimension. Which vector $\vec{w}$ deserves to be called " $\vec{u} - \vec{v}$ " ?

If $\vec{w} = $ " $\vec{u} - \vec{v}$ " then we should have

$$\vec{v} + \vec{w} = \vec{u} \quad , \quad yes \ ?$$
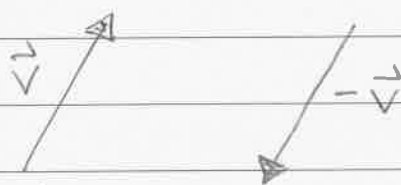
Picture!

So we could define $\vec{u} - \vec{v}$ as the arrow with tail at the point $\vec{v}$ and head at the point $\vec{u}$. One can check that this picture agrees with the algebraic rule

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} := \begin{pmatrix} u_1 - v_1 \\ u_2 - v_2 \\ \vdots \\ u_n - v_n \end{pmatrix}$$

But there is yet another way to think about subtraction of vectors:

Given an arrow $\vec{v}$ we will define the arrow "$-\vec{v}$" with the head and tail switched:

Then we have $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$.

Picture:
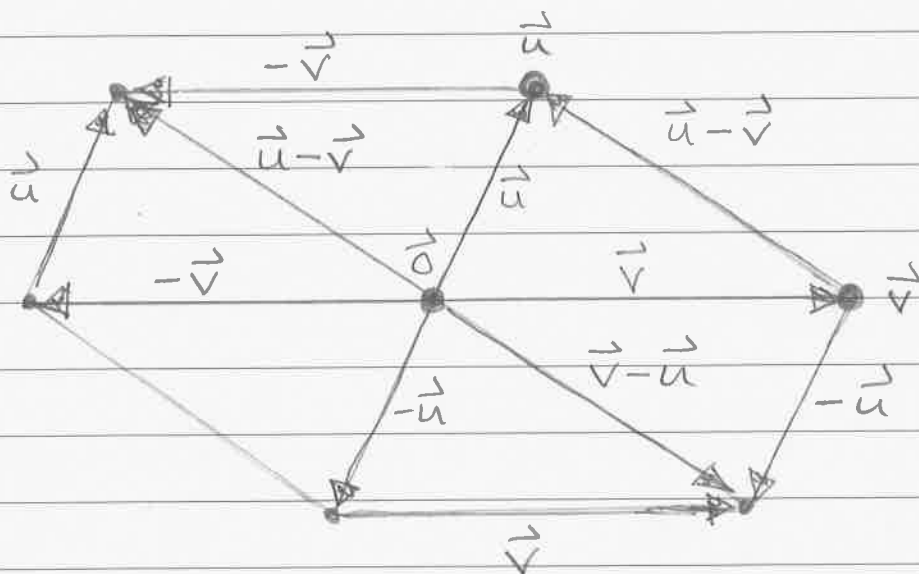
Everything fits together nicely. ☺

In summary, there are two ways to think about vectors

① Algebra of lists of numbers.
② Geometry of arrows.

The interaction between these two perspectives is what gives Linear Algebra its power.

The course webpage is up.

HW1 is due next Friday in class.

NO CLASS on Monday [MLK Day].

My Office Hours (Ungar 487):

- Mondays 12-1
- Thursdays 2-3
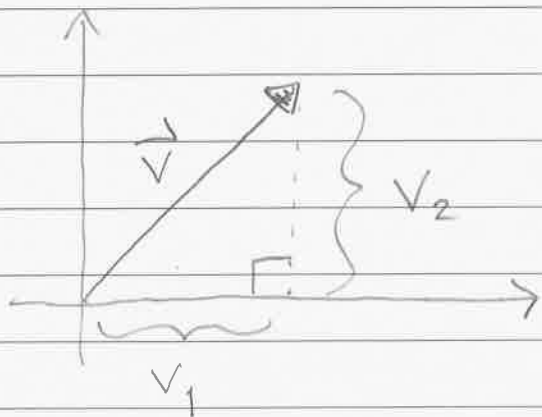- Also by Appointment.

We have seen that there are two ways to think about the vector

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

(1) The Cartesian coordinates of a point in "n-dimensional space".

(2) The arrow with tail at $(0, 0, \ldots, 0)$ and head at $(v_1, v_2, \ldots, v_n)$. We are allowed to move the arrow as long as we don't change its length or direction.

But what is the length of this vector?

Example ($n=2$): Let $\vec{v} = (v_1, v_2)$.

Let $\|\vec{v}\|$ denote the length of $\vec{v}$. Since the coordinate axes are perpendicular we have a right triangle and we can use the Pythagorean Theorem to get

$$\|\vec{v}\|^2 = v_1^2 + v_2^2$$

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}.$$

More generally, consider any two vectors $\vec{u} = (u_1, u_2)$ & $\vec{v} = (v_1, v_2)$ and their difference.

$$\vec{u} - \vec{v} = (u_1 - v_1, u_2 - v_2).$$

The length of the arrow $\vec{u} - \vec{v}$ is the
same as the distance between the points
$\vec{u}$ & $\vec{v}$, so we have

(*) $$dist(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$$
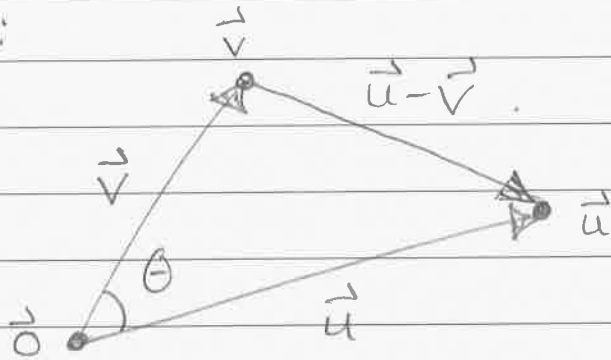
$$= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$$

Picture :
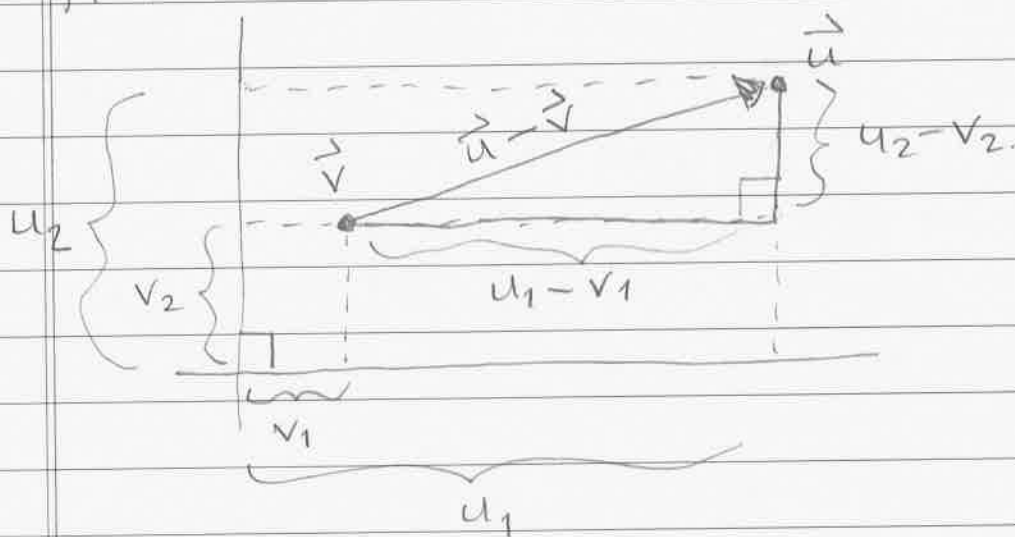
So formula (*) is just the Pythagorean
Theorem again.

But remember that there is another
useful way to think of the arrow
$\vec{u} - \vec{v}$: as the third side of a
triangle with arrows $\vec{u}$ & $\vec{v}$.

Picture:

Let θ be the angle between $\vec{u}$ & $\vec{v}$ when both of their tails are at $(0,0)$. Maybe you remember a fact about triangles called the Law of Cosines; it says that

$(**)$ $\qquad \|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\cdot\|\vec{v}\|\cos\theta.$

But now we have two expressions for the length/distance $\|\vec{u}-\vec{v}\|$ and a very interesting thing happens if we compare them.

From $(*)$ we have

$$\|\vec{u}-\vec{v}\|^2 = (u_1-v_1)^2 + (u_2-v_2)^2$$
$$= (u_1^2 - 2u_1v_1 + v_1^2) + (u_2^2 - 2u_2v_2 + v_2^2)$$
$$= (u_1^2 + u_2^2) + (v_1^2 + v_2^2) - 2(u_1v_1 + u_2v_2).$$
$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2(u_1v_1 + u_2v_2).$$

Then equating the two expressions
for $\|\vec{u} - \vec{v}\|^2$ gives

$$\|\vec{u}\|^2 + \|\vec{v}\|^2 - 2(u_1 v_1 + u_2 v_2) = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\| \cdot \|\vec{v}\| \cos\theta$$

$$-2(u_1 v_1 + u_2 v_2) = -2\|\vec{u}\| \cdot \|\vec{v}\| \cos\theta.$$

$$\boxed{u_1 v_1 + u_2 v_2 = \|\vec{u}\| \cdot \|\vec{v}\| \cos\theta}$$

This is a very funny formula. The
quantity on the left is new to us so
we need to give it a name.

☆ Definition : Given two n-dimensional
   vectors $\vec{u} = (u_1, \ , u_n)$ & $\vec{v} = (v_1, .., v_n)$
   we will define their dot product

$$\vec{u} \cdot \vec{v} := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

[Note that the dot product of two
 vectors is a number, not a vector.]

We have seen that the dot product in
two dimensions has the geometric
interpretation

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos\theta ,$$

algebra   geometry.

where θ is the angle between the
arrows $\vec{u}$ & $\vec{v}$ when their tails are at $\vec{0}$.
As a special case we have

$$\vec{u} \cdot \vec{u} = \|\vec{u}\| \cdot \|\vec{u}\| \cos 0$$
$$= \|\vec{u}\|^2 \cdot 1$$
$$= \|\vec{u}\|^2 .$$

So we can also express the length of
a vector in terms of the dot product

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} .$$

Q: Do the same formulas hold in
three-dimensional space ?

Consider the vector $\vec{v} = (v_1, v_2, v_3)$.
It is the diagonal of a rectangular box:
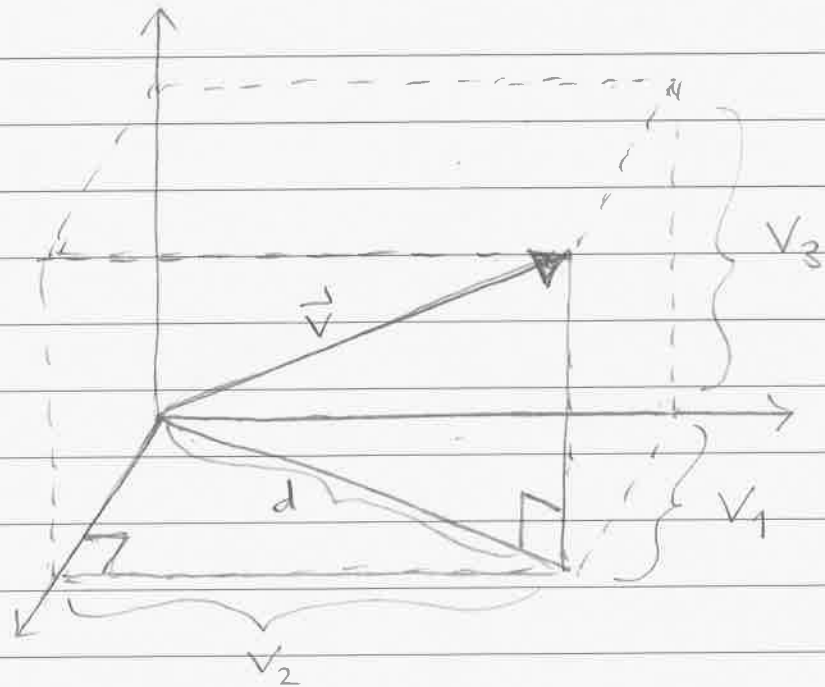
To compute the length $\|\vec{v}\|$ we will
use two right triangles:
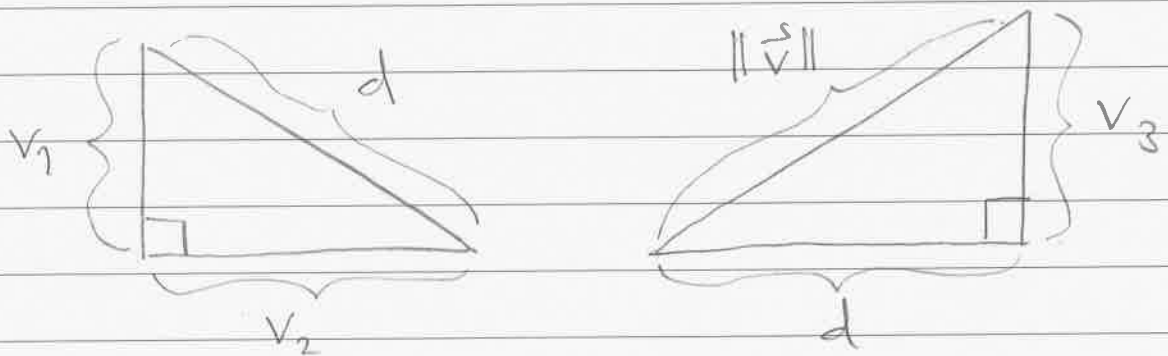
Applying the Pythagorean Theorem
to both triangles gives

$$d^2 = v_1^2 + v_2^2 \quad \& \quad \|\vec{v}\|^2 = d^2 + v_3^2$$

and hence

$$\|\vec{v}\|^2 = d^2 + v_3^2$$
$$= (v_1^2 + v_2^2)^2 + v_3^2$$
$$= v_1^2 + v_2^2 + v_3^2 .$$

We conclude that

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} . \qquad /\!/\!/$$

And this fact means that all of our
formulas from 2D are still true
in 3D.

Q: How about higher dimensions?
If $\vec{v} = (v_1, v_2, v_3, v_4)$, is it <u>true</u> that

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2} \quad ?$$

A: Sure, why not?

HW1 due Friday in class. No late homework will be accepted.

Note: The first meeting of the math club is this Thursday 6:30pm in Ungar 402.

Recall: Given two n-dimensional vectors

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \quad \& \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix},$$

their dot product is the number defined by

$$\vec{u} \circ \vec{v} := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

"vector $\circ$ vector = number"

The length of a vector satisfies

$$\|\vec{v}\|^2 = \vec{v} \circ \vec{v} = v_1^2 + v_2^2 + \cdots + v_n^2$$

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

When $n = 2$ or 3, we saw last time that this formula comes from the Pythagorean Theorem. For $n \geq 4$ we might as well use the formula $\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$ as the definition of "length".

Thinking Problem: What does $\|\vec{v}\|$ mean when $n = 1$?

Answer: A 1-dimensional vector is just a number, $\vec{v} = (v_1)$. In this case we might as well use the notation

$$\vec{v} = (v).$$

Then the formula for length gives

$$\|\vec{v}\| = \sqrt{v^2} = |v|$$

Thus the "length" of a vector generalizes the "absolute value" of a number.

Based on this, the distance between two points $\vec{u}$ & $\vec{v}$ in n-dimensional space is defined by

$$\text{dist}(\vec{u}, \vec{v}) := \|\vec{u} - \vec{v}\| = \sqrt{(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})}.$$

Using a purely algebraic argument gives

$(*)$
$$
\begin{aligned}
\|\vec{u} - \vec{v}\|^2 &= (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) \\
&= (\vec{u} - \vec{v}) \cdot \vec{u} - (\vec{u} - \vec{v}) \cdot \vec{v} \\
&= \vec{u} \cdot \vec{u} - \vec{v} \cdot \vec{u} - \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v} \\
&= \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - 2(\vec{u} \cdot \vec{v}) \\
&= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2(\vec{u} \cdot \vec{v})
\end{aligned}
$$

On the other hand, we can think of the vectors $\vec{u}, \vec{v}, \vec{u} - \vec{v}$ as forming a 2D triangle in n-dimensional space:
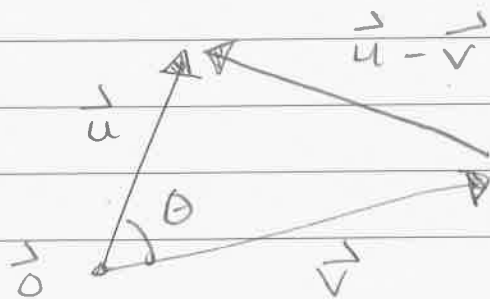
Then the Law of Cosines says that

$(**)$
$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\| \cdot \|\vec{v}\| \cos\theta$$

Finally, equating the expressions $(*)$ & $(**)$ for the number $\|\vec{u} - \vec{v}\|^2$ gives (after some simplification)
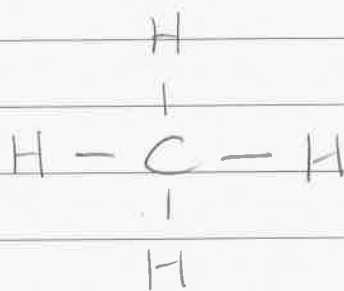
$$\vec{u} \circ \vec{v} = \| \vec{u} \| \cdot \| \vec{v} \| \cos \theta$$

Thus we have a geometric interpretation
for the dot product that holds in any
dimension.

Thinking Problem : How should you
interpret the boxed formula when $n=1$?

Application to Chemistry :

A molecule of methane consists of one carbon
atom surrounded by four hydrogen atoms:

$$H - C - H$$

with H above and below C:

```
        H
        |
   H  - C - H
        |
        H
```

But it doesn't look like this in real life;
for symmetry reasons it has the shape
of a "regular tetrahedron" with H's at
the vertices and C at the center.
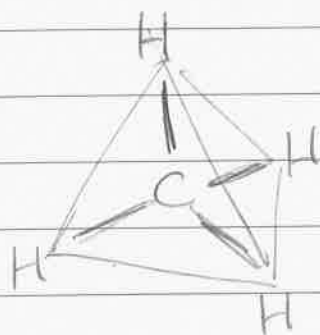
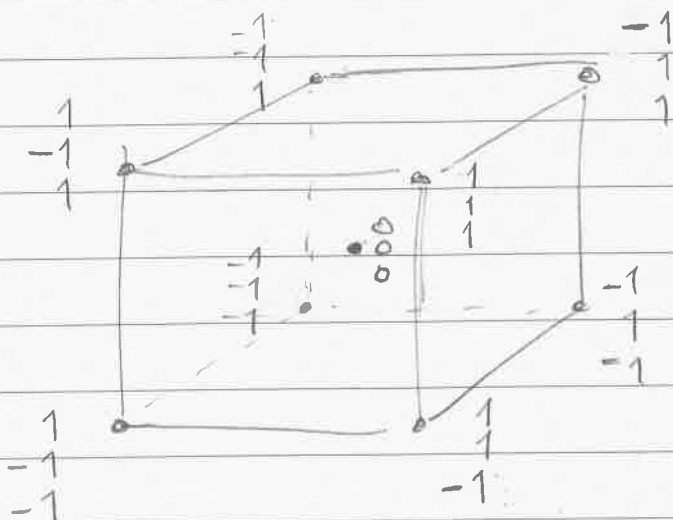Chemistry textbooks often say that the angle
between any two hydrogen atoms is

$$109.5°.$$

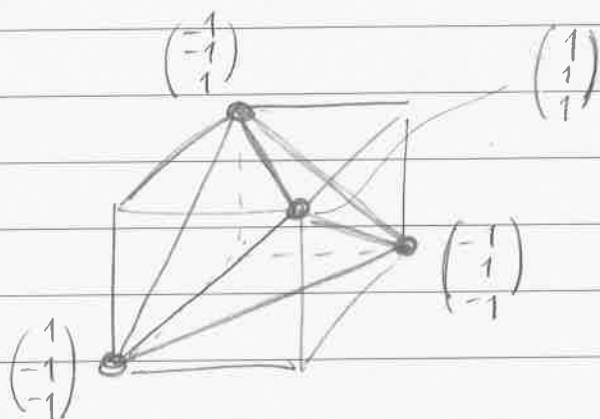But why is this? Let's use the dot product.

First we need a coordinate system.

Let's place C at the origin $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Now let's draw the cube whose coordinates
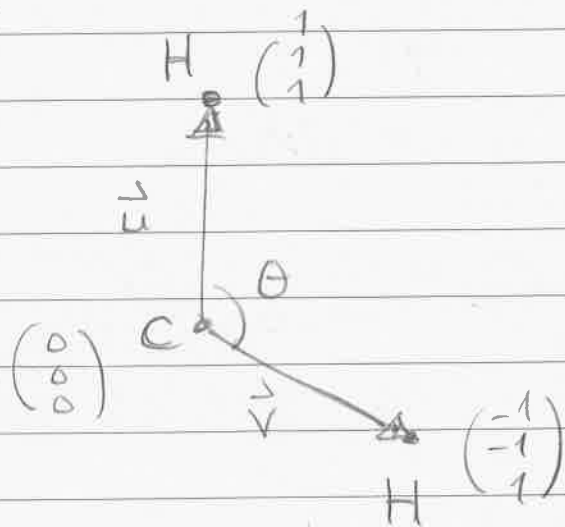are ± 1 (which is centered at $\vec{0}$):

[Compare to HW1(a).] The cube has 8
vertices. It turns out that if we choose
4 "alternating" vertices then they form the
vertices of a regular tetrahedron centered
at $\vec{0}$.

There are two choices; here's one:

$$\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

Finally, let's compute the angle between
any two vertices. There are six choices;
here's one:

H $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\vec{u}$

θ

$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ C

$\vec{v}$

$\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

H

We have

$$\|\vec{u}\|^2 = \vec{u} \circ \vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \circ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1^2 + 1^2 + 1^2 = 3$$

$$\|\vec{v}\|^2 = \vec{v} \circ \vec{v} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \circ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = (-1)^2 + (-1)^2 + 1^2 = 3.$$

$$\vec{u} \circ \vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \circ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = -1 - 1 + 1 = -1.$$

Then our boxed formula gives

$$\vec{u} \circ \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos\theta$$

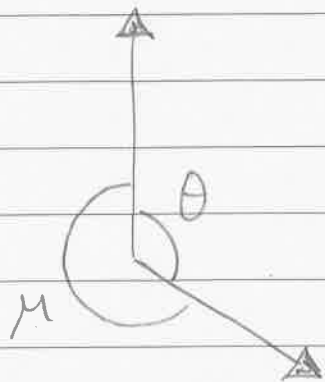$$-1 = \sqrt{3} \cdot \sqrt{3} \cos\theta .$$

$$-1 = 3 \cos\theta$$

$$-\frac{1}{3} = \cos\theta ,$$

and hence

$$\theta = \arccos\left(-\frac{1}{3}\right) \approx 109.47122^\circ .$$

Actually, there are two different angles with cosine $-1/3$. But there are also two different angles between the vectors:

Note that $\cos \mu = \cos \theta = -1/3$ where

$\theta \approx 109.5^\circ$ and $\mu \approx 360 - 109.5 = 250.5^\circ$

So everything works out nicely ☺.

Therefore it is reasonable to leave our answer in the form

$$\cos \theta = -1/3.$$