Math 210 F **Exam 2A**
Spring 2016 **Friday April 15**

This is a closed book test. No electronic devices are allowed. If two students submit exams in which any solution has been copied, **both students will receive a score of zero.** There are 5 pages and each page is worth 6 points, for a total of 30 points.

Problem 1. Let Π denote the following parametrized plane in $\mathbb{R}^3$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

(a) Let $\vec{a}$ be some vector that is perpendicular to the plane Π. Write down a single matrix equation to encode this information.

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix} \vec{a} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(b) Solve the matrix equation from part (a) to find all such vectors $\vec{a}$.

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 \end{array}\right) \longrightarrow \begin{array}{cccc} a & b & c & \\ \multicolumn{4}{c}{\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)} \end{array}.$$

If $\vec{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then we get

$$\vec{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -c \\ 0 \\ c \end{pmatrix} = c \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

(c) Use your answer from (b) to tell me an equation for Π.

$$-x + 0y + z = 0.$$

(d) Compute the matrix $P = \vec{a}(\vec{a}^T\vec{a})^{-1}\vec{a}^T$ that projects orthogonally onto the line $t\vec{a}$.

$$P = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \left(\begin{pmatrix} -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} (2)^{-1} \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

(e) Write down some matrix A whose column space is the plane Π.

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}.$$

(f) Finally, compute the matrix $Q = A(A^T A)^{-1}A^T$ that projects orthogonally onto the plane Π. [Hint: There is a shortcut using part (d).]

Since $P + Q = I$ we have

$$Q = I - P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Problem 2. Consider the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}.$$

(a) Let B be some matrix whose second **row** is $\begin{pmatrix} 1 & 2 & 1 \end{pmatrix}$. Compute the second **row** of the matrix BA

$$\begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 4 & 1 \end{pmatrix}$$

(b) Use Gaussian elimination to compute the inverse matrix A^{-1}.

$$(A \mid I) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 - 2R_2 \end{array}$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix}.$$

Problem 3. Consider a matrix A and two column vectors $\hat{x}, \vec{b}$ such that the following matrices exist: $(A^T A)^{-1}$, $A\hat{x}$, and $A^T\vec{b}$.

(a) If A has shape $m \times n$, tell me the shapes of $(A^T A)^{-1}$, $A\hat{x}$, and $A^T\vec{b}$.

$(A^T A)^{-1}$ has shape $n \times n$

$A\hat{x}$ has shape $m \times 1$

$A^T\vec{b}$ has shape $n \times 1$

(b) Now define the vector $\vec{e} := \vec{b} - A\hat{x}$. Tell me a single matrix equation that says that $\vec{e}$ is perpendicular to all of the columns of A.

$$A^T \vec{e} = \vec{0}$$

(c) Solve the matrix equation from (b) to find a formula for $\hat{x}$.

$$A^T(\vec{b} - A\hat{x}) = \vec{0}$$

$$A^T\vec{b} - A^T A\hat{x} = \vec{0}$$

$$(A^T A)\hat{x} = A^T\vec{b}$$

$$\cancel{(A^T A)^{-1}(A^T A)}\,\hat{x} = (A^T A)^{-1} A^T\vec{b}$$

$$\hat{x} = (A^T A)^{-1} A^T\vec{b}.$$

Problem 4. Consider the following three data points:

$$\begin{pmatrix} t \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(a) Write down a matrix equation that would be true if all three points were on the same line $C + tD = b$. This equation will have no solution.

$$\begin{cases} C + (-1)D = 2 \\ C + (0)D = 3 \\ C + (1)D = 1 \end{cases} \longrightarrow \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

(b) Write down the associated "normal equation", which does have a solution.

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

(c) Solve the normal equation to find the best fit line.

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 3C \\ 2D \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix} \rightarrow \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 2 \\ -1/2 \end{pmatrix}.$$

The best fit line is

$$2 - \frac{1}{2}t = b.$$