Math 210 F
Spring 2016

Exam 1A
Friday March 3

This is a closed book test. No electronic devices are allowed. If two students submit exams in which any solution has been copied, **both students will receive a score of zero.** There are 5 pages and 5 problems, each worth 6 points.

Problem 1. Let $\vec{x}$ and $\vec{y}$ be two vectors (in some-dimensional space) such that

$$\|\vec{x}\| = 2, \qquad \|\vec{y}\| = \sqrt{2}, \qquad \text{and} \qquad \vec{x} \bullet \vec{y} = 2.$$

(a) Find the **cosine** of the angle between $\vec{x}$ and $\vec{y}$ (and the angle itself, if you know it).

$$\|\vec{x}\| \, \|\vec{y}\| \cos\theta = \vec{x} \circ \vec{y}$$

$$\cos\theta = (\vec{x} \circ \vec{y}) / (\|\vec{x}\| \, \|\vec{y}\|)$$

$$= 2/(2\sqrt{2}) = 1/\sqrt{2} \implies \theta = \pm \frac{\pi}{4}$$

(b) Tell me the values of the dot products $\vec{x} \bullet \vec{x}$ and $\vec{y} \bullet \vec{y}$.

$$\vec{x} \circ \vec{x} = \|\vec{x}\|^2 = 2^2 = 4$$

$$\vec{y} \circ \vec{y} = \|\vec{y}\|^2 = (\sqrt{2})^2 = 2$$

(c) Expand the expression $(\vec{y} - \vec{x}) \bullet (\vec{y} - \vec{x})$ and use the result to find the **distance between the points two points** $\vec{x}$ and $\vec{y}$.

$$\|\vec{y} - \vec{x}\|^2 = (\vec{y} - \vec{x}) \circ (\vec{y} - \vec{x})$$

$$= \vec{y} \circ \vec{y} - 2(\vec{x} \circ \vec{y}) + \vec{x} \circ \vec{x}$$

$$= 4 - 2(2) + 2$$

$$= 2$$

$$\implies \|\vec{y} - \vec{x}\| = \sqrt{2}.$$

Problem 2. Consider the following system of 3 linear equations in 3 unknowns:

$$\begin{cases} x + y + 2z = 0 \\ x + 2y + 3z = 1 \\ 0 + y + z = 1 \end{cases}$$

(a) Put the system in reduced row echelon form (RREF).

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right)\begin{array}{c} A \\ B \\ C \end{array} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right)\begin{array}{c} A \\ B-A \\ C \end{array}$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)\begin{array}{c} A \\ B \\ C-B \end{array} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)\begin{array}{c} A-B \\ B \\ C \end{array}$$

$$\begin{cases} x + 0 + z = -1 \\ y + z = 1 \\ 0 = 0 \end{cases}$$

(b) Use your answer from part (a) to write out the **complete solution** of the system.

Let $z = t$. Then we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1-t \\ 1-t \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}}_{\text{This is a line in } \mathbb{R}^3}$$

(c) Fill in the blanks: Geometrically, this system represents three planes that intersect at a Line.

Problem 3. Now consider the modified system of 3 linear equations in 3 unknowns, where c is an arbitrary constant:

$$\begin{cases} x + \ \ y + 2z = 0 \\ x + 2y + 3z = 1 \\ 0 + \ \ y + \ \ z = c \end{cases}$$

(a) Put the system in upper-staircase form (you don't need to put it in RREF).

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & c \end{array}\right) \begin{array}{c} A \\ B \\ C \end{array} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & c \end{array}\right) \begin{array}{c} A \\ B-A \\ C \end{array}$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & c-1 \end{array}\right) \begin{array}{c} A \\ B \\ C-B \end{array} \qquad \begin{cases} x+y+2z = 0 \\ y+z = 1 \\ 0 = c-1. \end{cases}$$

(b) Use part (a) to find all values of c such that the system has **no solution**.

If $c \neq 1$ then the equation $0 = c-1$ is false, so the system has no solution

(c) If $c = 1$ then we already saw in Problem 2 that the system **has a solution**. Use your solution to express the vector $(0, 1, 1)$ as a specific linear combination of the vectors $(1, 1, 0)$, $(1, 2, 1)$, and $(2, 3, 1)$.

For all t we have

$$(-1-t)\begin{pmatrix}1\\1\\0\end{pmatrix} + (1-t)\begin{pmatrix}1\\2\\1\end{pmatrix} + t\begin{pmatrix}2\\3\\1\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix}$$

Now choose any t you want.

Problem 4. Let A be an $m \times n$ matrix (i.e. with m rows and n columns) and let B be a $p \times q$ matrix (i.e. with p rows and q columns).

(a) Fill in the blanks:

We think of A as a function from $\underline{n}$-dimensional space to $\underline{m}$-dimensional space.

We think of B as a function from $\underline{q}$-dimensional space to $\underline{p}$-dimensional space.

$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m, \quad \mathbb{R}^q \xrightarrow{B} \mathbb{R}^p$$

(b) Finish the sentence: The product matrix AB is defined only when ...

columns of A = # rows of B

$$n = p$$

(c) Fill in the blanks: If the product matrix AB is defined then we think of it as a function from $\underline{q}$-dimensional space to $\underline{m}$-dimensional space.

$$\begin{tikzcd} \mathbb{R}^q \arrow[r, "B"] \arrow[rr, bend right, "AB"'] & \underset{\mathbb{R}^p}{\mathbb{R}^n} \arrow[r, "A"] & \mathbb{R}^m \end{tikzcd}$$

(d) Finish the sentence: If the matrix AB is defined then its entry in the ith row and jth column is equal to ...

(ith row of A) $\cdot$ (jth col. of B)

Problem 5. Consider the following two matrices and one vector:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \qquad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

(a) Find the vector $B\vec{x}$ by computing the dot product of $\vec{x}$ with the rows of B.

$$B\vec{x} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} (1\ -1\ 0)\begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ (0\ 2\ 1)\begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{pmatrix} = \begin{pmatrix} x - y \\ 2y + z \end{pmatrix}.$$

(b) Express the vector $A(B\vec{x})$ as a linear combination of the columns of A.

$$A(B\vec{x}) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x - y \\ 2y + z \end{pmatrix}$$

$$= (x-y)\begin{pmatrix} 1 \\ 3 \end{pmatrix} + (2y+z)\begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} x - y \\ 3x - 3y \end{pmatrix} + \begin{pmatrix} 4y + 2z \\ 8y + 4z \end{pmatrix} = \begin{pmatrix} x + 3y + 2z \\ 3x + 5y + 4z \end{pmatrix}$$

(c) Now find the matrix C such that for all numbers x, y, z we have $C\vec{x} = A(B\vec{x})$.

$$C\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 3y + 2z \\ 3x + 5y + 4z \end{pmatrix}$$

$$\Longrightarrow \quad C = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 5 & 4 \end{pmatrix}.$$