Friday Apr 5.

HW 8 due Monday.

Now: E.values & E.vectors

Consider the matrix $A = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}$.

Using a computer we find

$$A^2 = \begin{pmatrix} .70 & .45 \\ .30 & .55 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} .650 & .525 \\ .350 & .475 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} .6250 & 0.5622 \\ .3750 & 0.4375 \end{pmatrix}$$

$\vdots$

$$A^{10} = \begin{pmatrix} 0.600 & 0.599 \\ 0.399 & 0.401 \end{pmatrix}$$

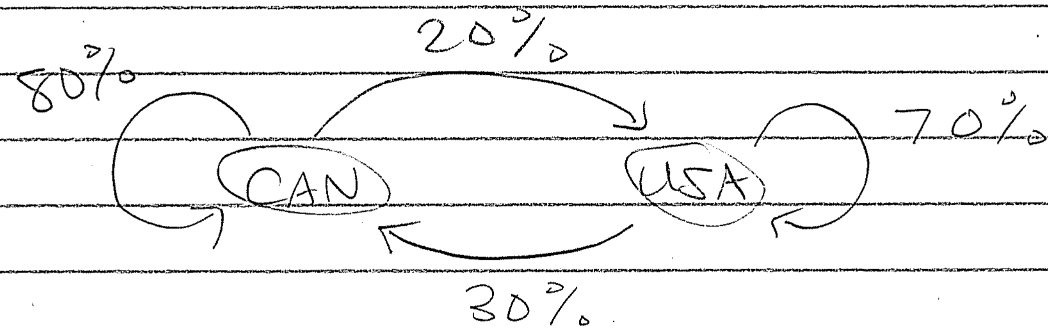
It seems like

$$\lim_{n \to \infty} A^n = \begin{pmatrix} .6 & .6 \\ .4 & .4 \end{pmatrix}$$

What's going on here?

=

Consider a simple model. A species of bird lives in Canada and the USA. Every year there is a migration



Assume no birds are born or die.
In year n there are

c_n birds in CAN.
u_n birds in USA

How are $\begin{pmatrix} c_n \\ u_n \end{pmatrix}$ and $\begin{pmatrix} c_{n+1} \\ u_{n+1} \end{pmatrix}$ related?

Of the c_n birds in CAN now, $.8c_n$ stay and $.2c_n$ move. Of the u_n birds in USA now, $.7u_n$ stay and $.3u_n$ move.

Hence

$$c_{n+1} = .8c_n + .3u_n$$
$$u_{n+1} = .2u_n + .7u_n$$

i.e.
$$\begin{pmatrix} c_{n+1} \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix} \begin{pmatrix} c_n \\ u_n \end{pmatrix}$$

$$\vec{v}_{n+1} = A\vec{v}_n$$

Say $\vec{v}_n$ is the state vector at time n

Say A is the transition matrix.

Example

Start with $\vec{v}_0 = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$

Then $\vec{v}_1 = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}\begin{pmatrix} 10 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$

$$\vec{v}_2 = A\vec{v}_1 = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}\begin{pmatrix} 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

$$\vec{v}_3 = A\vec{v}_2 = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}\begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 6.5 \\ 3.5 \end{pmatrix}$$

Q: 6.5 birds?
A: Yes. We're just dealing with probabilities.

In general we have

$$\begin{aligned}
\vec{v}_n &= A\vec{v}_{n-1} \\
&= AA\vec{v}_{n-2} \\
&= AAA\vec{v}_{n-3} \\
&\;\vdots \\
&= \underbrace{AAA\cdots A}_{n \text{ times}}\vec{v}_0 \\
&= A^n\vec{v}_0 = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}^n \begin{pmatrix} 10 \\ 0 \end{pmatrix}
\end{aligned}$$

Can we compute this?

Big Idea: We will say state $\vec{v}$ is an equilibrium of the system if

$$A\vec{v} = \vec{v} = 1\vec{v}.$$

An eigenvector with eigenvalue 1

If it exists, let's compute it.

Let $\vec{v} = \begin{pmatrix} c \\ u \end{pmatrix}$.

Then $A\vec{v} = \vec{v}$.

$$\Rightarrow \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix} \begin{pmatrix} c \\ u \end{pmatrix} = \begin{pmatrix} c \\ u \end{pmatrix}.$$

$$\Rightarrow \begin{aligned} .8c + .3u &= c \\ .2c + .7u &= u. \end{aligned}$$

$$\Rightarrow \begin{aligned} -.2c + .3u &= 0 \\ \cancel{.2c - .3u = 0}. \end{aligned}$$ redundant.

$$\Rightarrow \begin{aligned} .3u &= .2c \\ 3u &= 2c. \end{aligned}$$

$$\Rightarrow u/c = 2/3$$

The 1-eigenspace of A is the line

$$\begin{pmatrix} c \\ u \end{pmatrix} = t\begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

In particular we have

$$A\begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}.$$

6 birds in CAN
4 birds in USA is an equilibrium.

But we haven't yet explained why

$$A^n\begin{pmatrix} 10 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

as $n \longrightarrow \infty$

To do this we need the <u>other</u> eigenvalue.

The characteristic equation of $\begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}$ is

$$(.8-\lambda)(.7-\lambda)-(.2)(.3)=0$$
$$.56-.8\lambda-.7\lambda+\lambda^2-.06=0$$

$$\lambda^2-1.5\lambda+.5=0$$
$$2\lambda^2-3\lambda+1=0$$

Hence the eigenvalues are

$$\lambda=\frac{3\pm\sqrt{(-3)^2-4(1)(2)}}{2(2)}=\frac{3\pm 1}{4}$$

$$=1 \text{ or } .5$$

Let's compute the eigenvalues corresponding to eigenvalue .5

$$\begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}\begin{pmatrix} c \\ u \end{pmatrix}=.5\begin{pmatrix} c \\ u \end{pmatrix}$$

$$\Longrightarrow \quad .8c+.3u=.5c$$
$$.2c+.7u=.5u$$

$$\Longrightarrow \quad .3c + .3u = 0$$

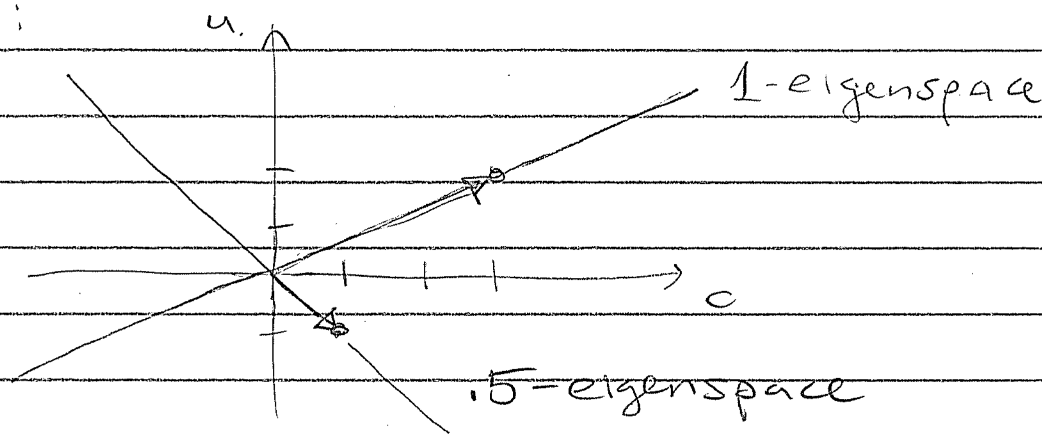
$\cancel{.2c + .2u = 0}$ Redundant ☺

$$\Longrightarrow \quad c + u = 0.$$

So the ".5-eigenspace" is the line

$$\begin{pmatrix} c \\ u \end{pmatrix} = t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

<u>Picture</u>:



Slogan: Once you know the eigenvectors, you should express everything in terms of them.

For example, let's express our initial state vector:

$$\begin{pmatrix} 10 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 2 \end{pmatrix} + 4\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Then we have

$$A^n \begin{pmatrix} 10 \\ 0 \end{pmatrix} = A^n \left[2\begin{pmatrix} 3 \\ 2 \end{pmatrix} + 4\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right]$$

$$= 2A^n \begin{pmatrix} 3 \\ 2 \end{pmatrix} + 4A^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= 2\begin{pmatrix} 3 \\ 2 \end{pmatrix} + 4\left(\frac{1}{2}\right)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 + 4/2^n \\ 4 - 4/2^n \end{pmatrix}$$

As $n \to \infty$ we have

$$A^n \begin{pmatrix} 10 \\ 0 \end{pmatrix} \to \begin{pmatrix} 6 + 0 \\ 4 + 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

Mon Apr 8

HW 8 due NOW

HW 9 due next Mon. Apr. 15

Today: HW 8 Discussion

6.1.9. Assume that x is an e.vector of A with e.value λ.

(a) Then x is <u>also</u> an e.vector of A^2, but with e.value λ^2.

<u>Proof</u>: We have

$$A^2\vec{x} = A(A\vec{x}) = A(\lambda\vec{x})$$
$$= \lambda A\vec{x} = \lambda\lambda\vec{x} = = \lambda^2\vec{x}$$

$\blacksquare$

(b) Then x is <u>also</u> an e.vector of A^{-1} (if A^{-1} exists), but with e.value $\lambda^{-1} \left(= \frac{1}{\lambda}\right)$.

<u>Proof</u>: We have

$$\vec{x} = I\vec{x} = (A^{-1}A)\vec{x} = A^{-1}(A\vec{x})$$
$$= A^{-1}(\lambda\vec{x})$$
$$= \lambda(A^{-1}\vec{x}).$$

Hence

$$A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}. \qquad \blacksquare$$

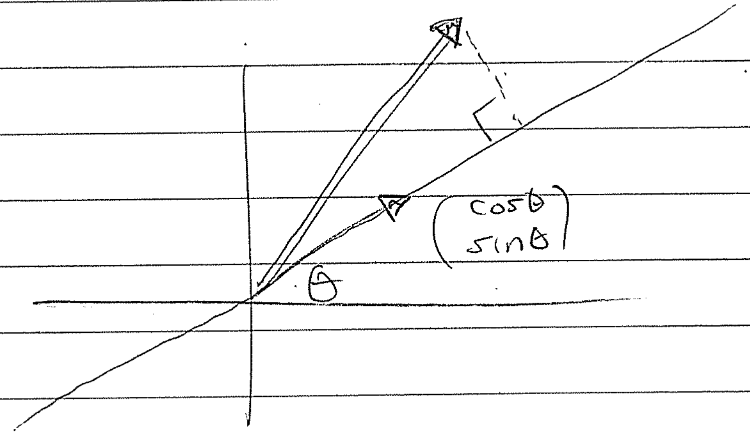
(c) Then x is <u>also</u> an e.vector of $A+I$ but with e.value $\lambda+1$.

<u>Proof</u>: We have

$$\begin{aligned}(A+I)\vec{x} &= A\vec{x} + I\vec{x}\\ &= \lambda\vec{x} + 1\vec{x}\\ &= (\lambda+1)\vec{x} \qquad \blacksquare\end{aligned}$$

Problem A.1.

Let P be the matrix that projects onto the line through $(\cos\theta, \sin\theta)$.



To save space, let's write $c = \cos\theta$ and
$s = \sin\theta$.

The projection matrix is $P = \frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}}$ where $\vec{a} = \begin{pmatrix} c \\ s \end{pmatrix}$

$$\Longrightarrow \quad P = \frac{\begin{pmatrix} c \\ s \end{pmatrix}(c \; s)}{(c \; s)\begin{pmatrix} c \\ s \end{pmatrix}} = \frac{1}{c^2+s^2}\begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}$$

$$= \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}$$

because $c^2+s^2 = 1$, as you know.

The e.values of P are given by

$$(c^2-\lambda)(s^2-\lambda) - (cs)(cs) = 0.$$

$$\cancel{c^2s^2} - c^2\lambda - s^2\lambda + \lambda^2 - \cancel{c^2s^2} = 0$$

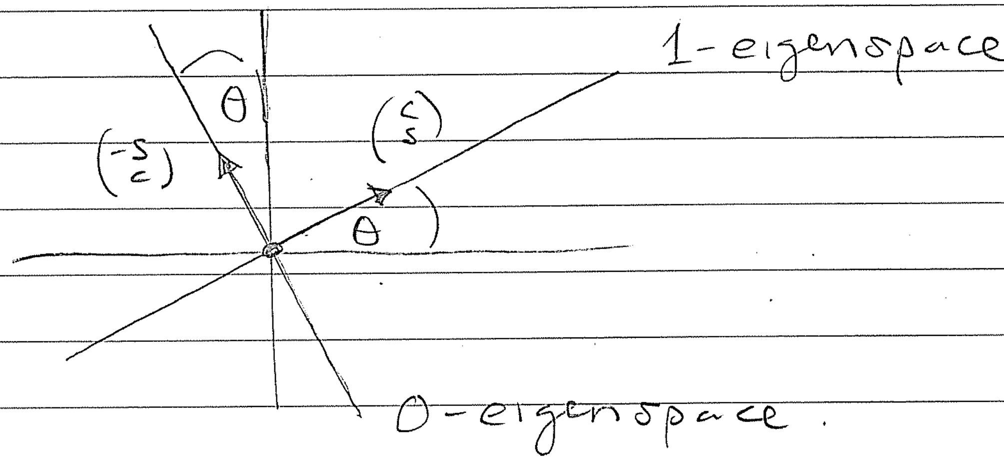
$$\lambda^2 - (c^2+s^2)\lambda = 0.$$

$$\lambda^2 - \lambda = 0$$

$$\lambda(\lambda - 1) = 0$$

$\Longrightarrow$ E.values are $\lambda = 1$ and 0.

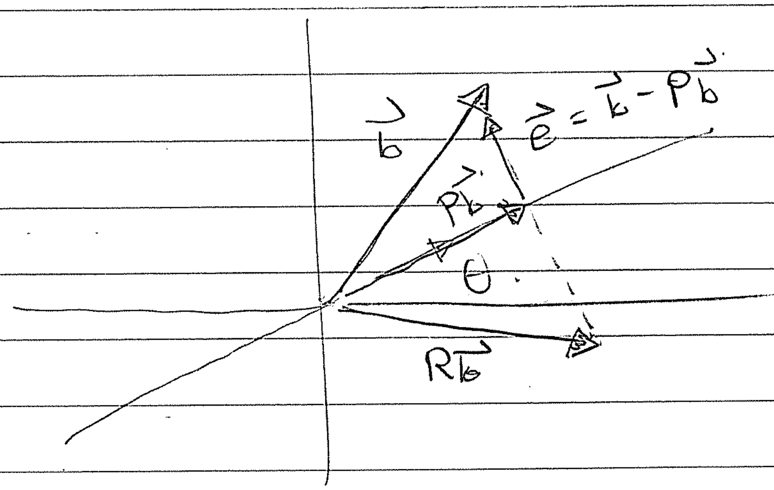
E.vectors ? Claim:



You can easily check this.

Problem A.2.

Q: What is the matrix of the reflection accross the line through $(\cos\theta, \sin\theta)$?



Note that

$$R\vec{b} = \vec{b} - 2\vec{e} = \vec{b} - 2(\vec{b} - P\vec{b})$$
$$= 2P\vec{b} - \vec{b}$$
$$= 2P\vec{b} - I\vec{b}$$
$$= (2P - I)\,\vec{b}$$

$$\Longrightarrow \quad R = 2P - I.$$

$$= 2\begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2c^2 - 1 & 2cs \\ 2cs & 2s^2 - 1 \end{pmatrix}$$

E. values of R? Easy.

Suppose $P\vec{x} = \lambda\vec{x}$. Then

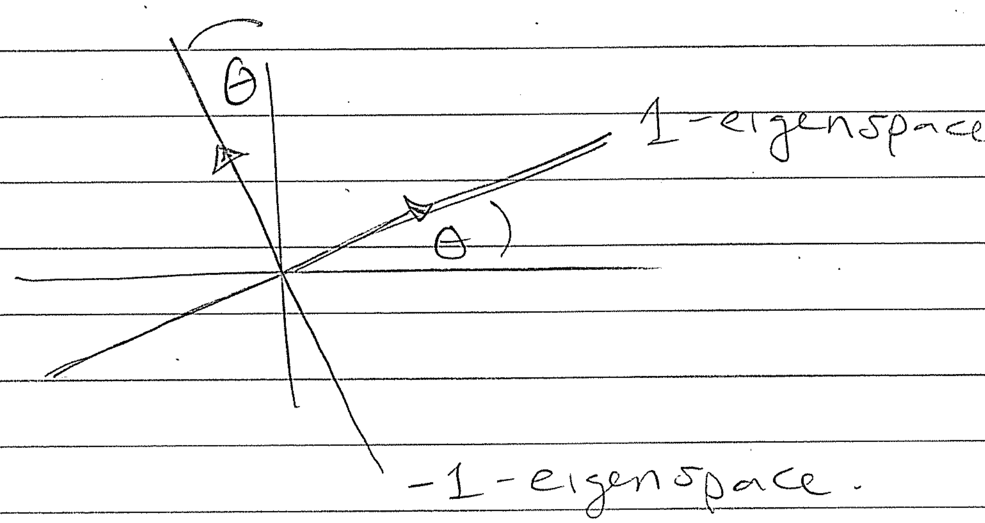
$$R\vec{x} = (2P - I)\,\vec{x}$$
$$= 2P\vec{x} - I\vec{x}$$
$$= 2\lambda\vec{x} - \vec{x}$$
$$= (2\lambda - 1)\,\vec{x}.$$

$\Rightarrow$ E.vectors of R the same as for P, but the E.values have changed from λ to $2\lambda - 1$.

P has evalues 1 and 0.

R has evalues $2(1)-1$ and $2(0)-1$
1 and -1.

Picture:



Problem 6.1.14. Find the E.values of the rotation matrix

$$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Characteristic Equation:

$$(c-\lambda)(c-\lambda) - s(-s) = 0.$$

$$c^2 - 2c\lambda + \lambda^2 + s^2 = 0.$$

$$\lambda^2 - 2c\lambda + (c^2+s^2) = 0$$

$$\lambda - 2c\lambda + 1 = 0.$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \frac{2\cos\theta \pm \sqrt{4(\cos^2\theta - 1)}}{2}.$$

$$= \frac{2\cos\theta \pm \sqrt{4(-\sin^2\theta)}}{2}.$$

$$= \frac{2\cos\theta \pm 2\sqrt{-1}\sin\theta}{2}.$$

$$= \cos\theta \pm \sqrt{-1}\sin\theta.$$

NO REAL Eigenvalues unless $\sin\theta = 0$

This is the meaning of complex eigenvalues:

If 2x2 matrix A has complex eigenvalues, then it has a tendency to **rotate**.

If A is the transition matrix of a dynamical system, then the system will **oscillate**.

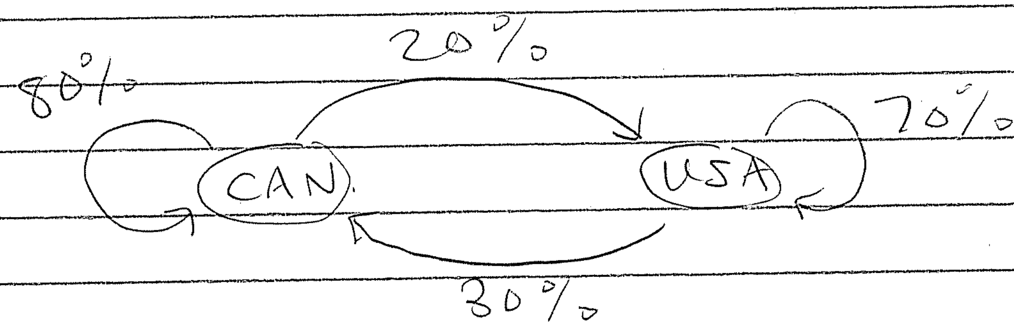
Wed Apr 10

HW 9 due MON Apr 15
Exam 2 FRI Apr 19.
Office Hours Today 3–4.

Math Club Today 5:30 pm
in Ungar 402
I will do a magic trick.

Today: Phase Portraits.

Recall the birds



and their transition matrix

$$A = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}$$

If we let c_n = # birds in CAN at year n
u_n = # birds in USA at year n.

Then we have.

$$\begin{pmatrix} c_n \\ u_n \end{pmatrix} = A \begin{pmatrix} c_{n-1} \\ u_{n-1} \end{pmatrix}$$

$$= A\,A \begin{pmatrix} c_{n-2} \\ u_{n-2} \end{pmatrix}$$

$$\vdots$$

$$= \underbrace{A\,A \cdots A}_{n \text{ times}} \begin{pmatrix} c_0 \\ u_0 \end{pmatrix}$$

$$= A^n \begin{pmatrix} c_0 \\ u_0 \end{pmatrix}$$

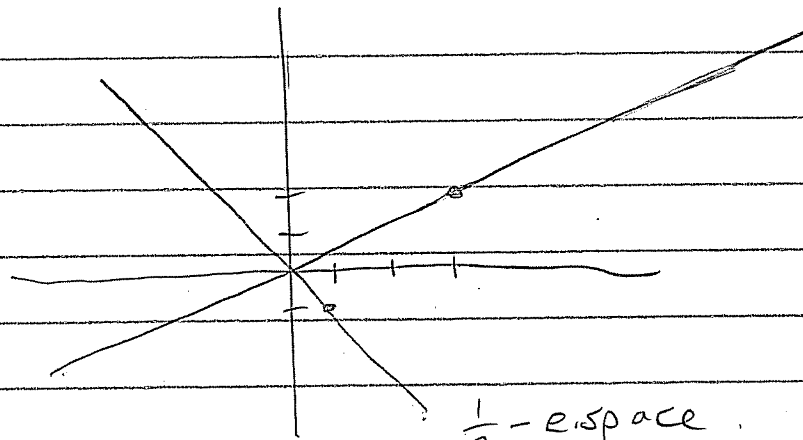
To solve this system, i.e., to find formulas for (c_n, u_n) in terms of (c_0, u_0), we must compute the eigenvalues / eigenvectors.

The e.values are 1 and $.5$.

The e.vectors are.

$$A t \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 1 \cdot t \begin{pmatrix} 3 \\ 2 \end{pmatrix} \;\&\; A t \begin{pmatrix} 1 \\ -1 \end{pmatrix} = .5 \cdot t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Picture:



This picture tells us a **lot**.
Suppose we start with $(c_0, u_0) = (10, 0)$.
This can be written in terms of eigenvectors as

$$\begin{pmatrix} 10 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 2 \end{pmatrix} + 4\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

Then we have

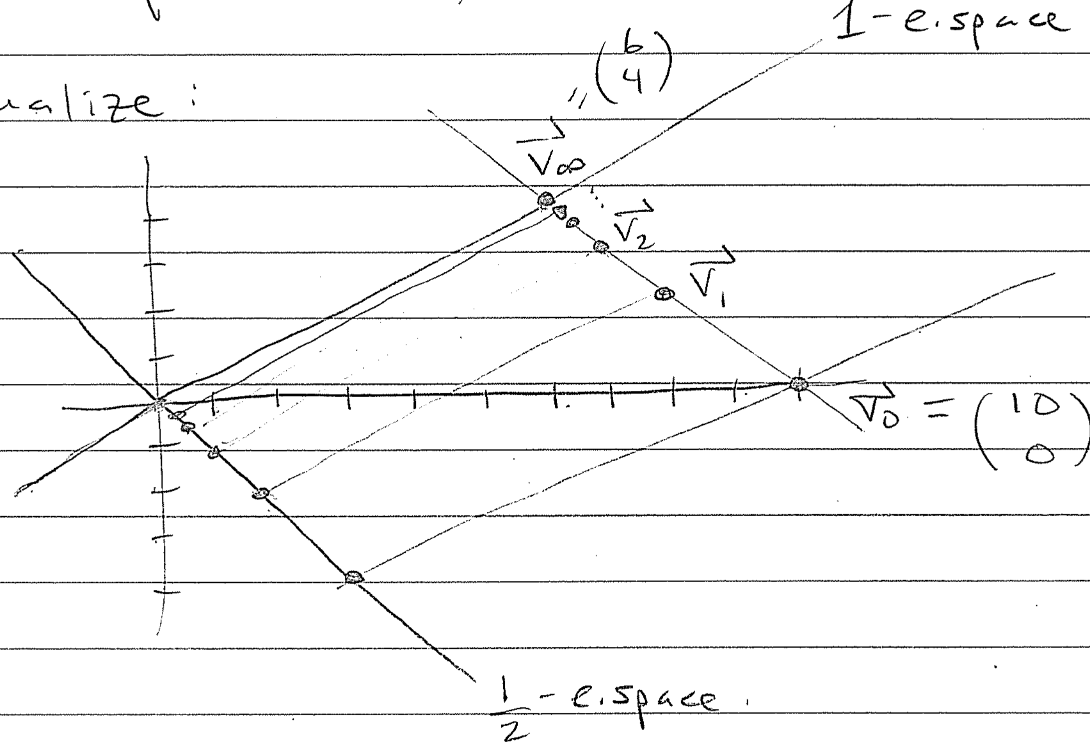
$$\begin{pmatrix} c_n \\ u_n \end{pmatrix} = A^n \begin{pmatrix} 10 \\ 0 \end{pmatrix} = A^n \left[\begin{pmatrix} 6 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ -4 \end{pmatrix} \right]$$

$$= A^n \begin{pmatrix} 6 \\ 4 \end{pmatrix} + A^n \begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

$$= 1^n \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \left(\frac{1}{2}\right)^n \begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

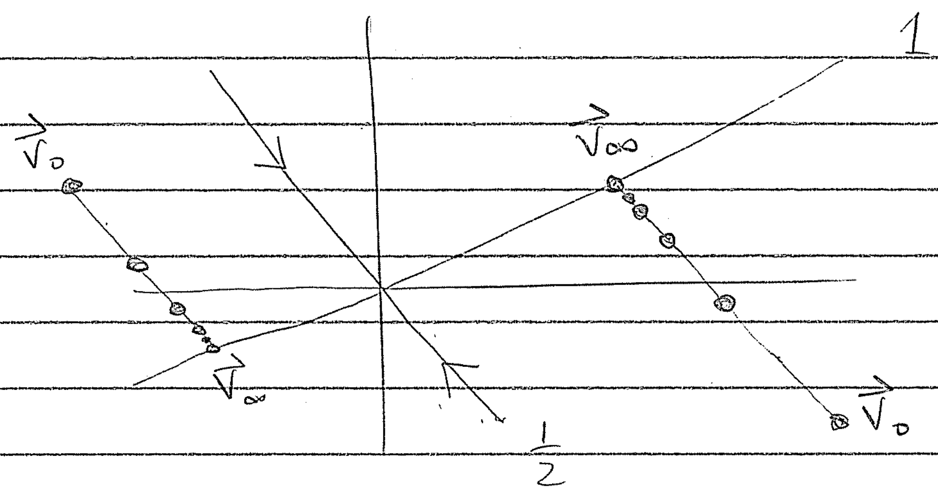
$$= \begin{pmatrix} 6 + 4/2^n \\ 4 - 4/2^n \end{pmatrix}$$

Visualize:



At each step, the state halves in the (1,-1) direction and stays the same in the (3,2) direction.

A general trajectory.



So the matrix A^∞ is a projection onto the line $t(3,2)$, but at a strange angle (i.e. not $90°$).

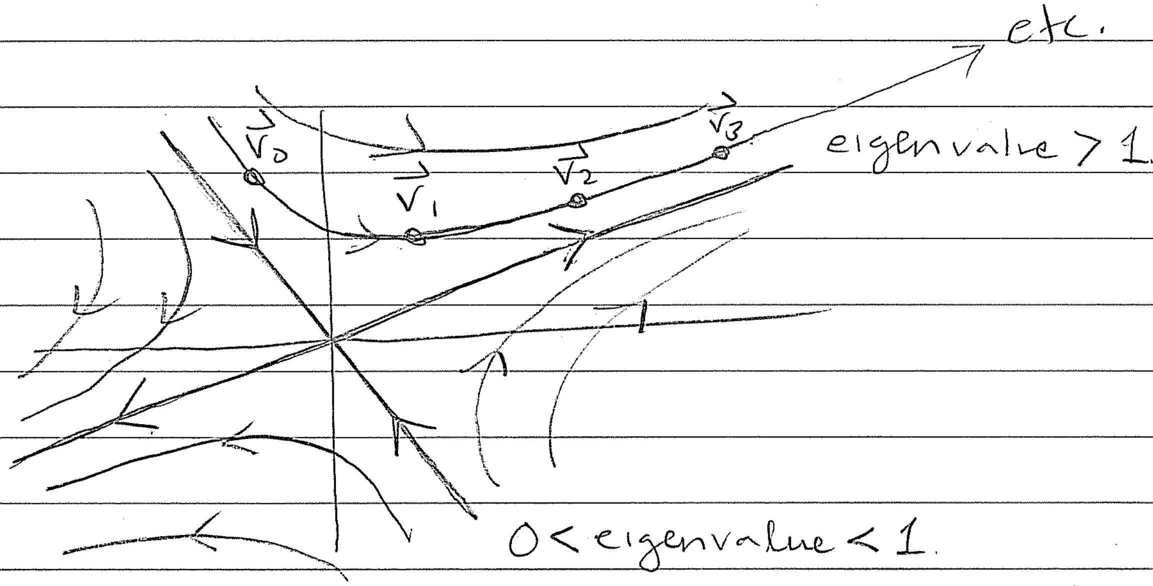
In fact, $A^\infty = \begin{pmatrix} .6 & .6 \\ .4 & .4 \end{pmatrix}$

The orthogonal projection would be

$$P = \frac{\begin{pmatrix} 3 \\ 2 \end{pmatrix}(3\ 2)}{(3\ 2)\begin{pmatrix} 3 \\ 2 \end{pmatrix}} = \frac{1}{13}\begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix} \neq A^\infty.$$

Note: A^∞ has the same e.vectors but with e.values $1^\infty = \underline{1}$ and $(.5)^\infty = 0$

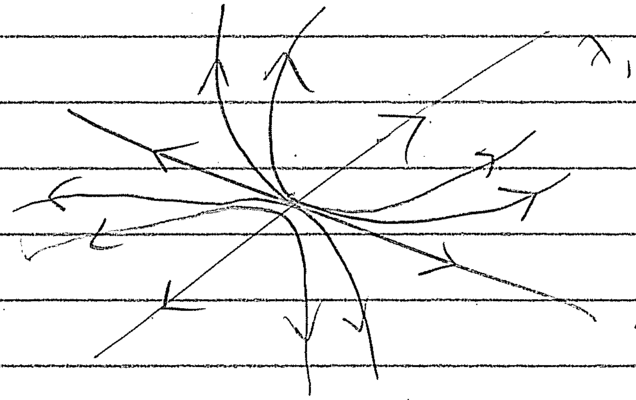
Q: What if we had a matrix with <u>this</u> eigen-information:



This is called a "phase portrait"
It shows us the <u>typical</u> <u>trajectories</u>.

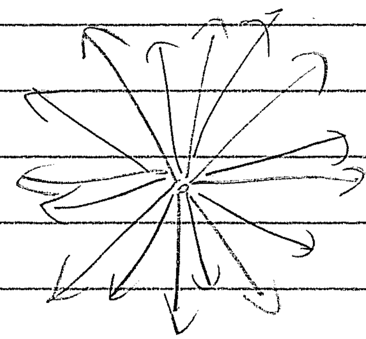
The evalues/evectors determine the behavior of the system.

Other Possibilities: $\boxed{\lambda_1 > \lambda_2 > 1}$

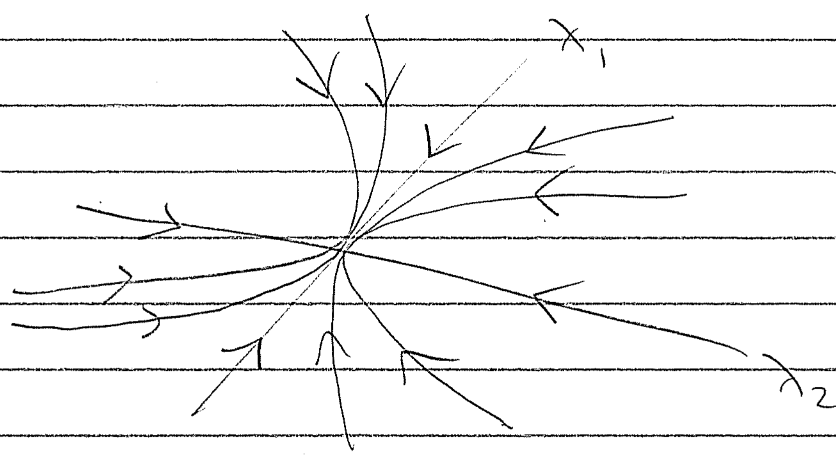


$\boxed{\lambda_1 = \lambda_2 > 1}$

Expands evenly in all directions



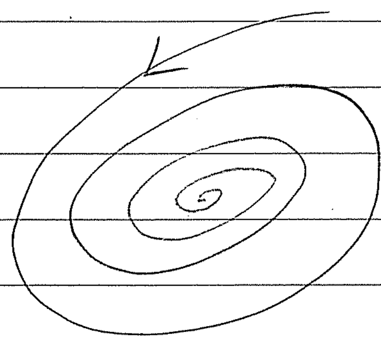
$\boxed{0 < \lambda_1 < \lambda_2 < 1}$



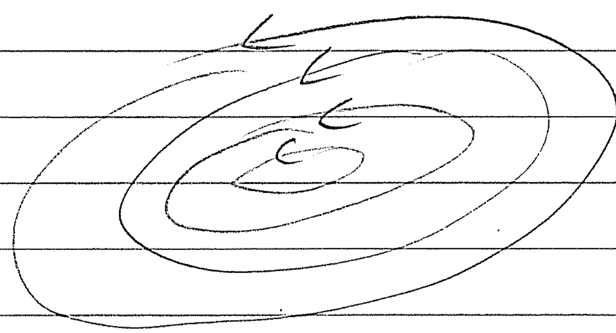
What if λ_1, λ_2 are complex?

Then the system will oscillate

$|\lambda_1| = |\lambda_2| < 1$

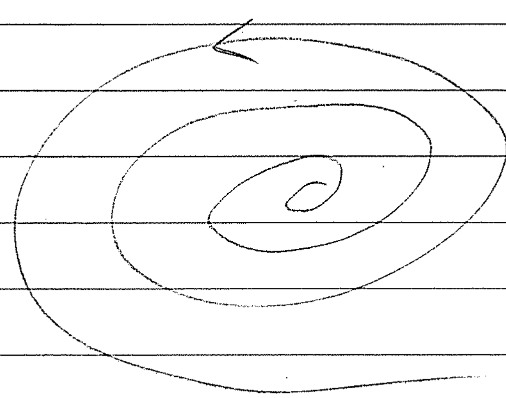


$|\lambda_1| = |\lambda_2| = 1$



closed orbits

$|\lambda_1| = |\lambda_2| > 1$



Fri Apr 12

HW 9 due Mon
Exam 2 next Fri

Today: The phase portrait of Fibonacci numbers.

Recall the Fibonacci numbers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

They are defined by initial conditions

$$\begin{pmatrix} F_1 \\ F_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and second-order recurrence

$$F_{n+2} = F_{n+1} + F_n \quad \text{for all } n \geq 0.$$

which we can write as a 2×2 matrix equation (dynamical system)

$$\begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$$

If we let $\vec{V}_n = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$ then we can translate this as:

Initial condition $\vec{V}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Recurrence $\vec{V}_{n+1} = A\vec{V}_n$ for all $n \geq 0$,

where $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

<u>Problem</u>: <u>Solve</u> the system.

Compute eigenvalues.

$$(1-\lambda)(0-\lambda) - 1\cdot 1 = 0$$
$$-\lambda + \lambda^2 - 1 = 0$$
$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{(-1)^2 - 4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Call these $\alpha = \frac{1+\sqrt{5}}{2}$

$$\beta = \frac{1-\sqrt{5}}{2}$$

and observe that $\alpha^2 - \alpha - 1 = 0$

$$\beta^2 - \beta - 1 = 0$$
$$\alpha + \beta = 1$$
$$\alpha\beta = -1$$

Compute eigenvectors:

$\lambda = \alpha$: $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \alpha\begin{pmatrix} x \\ y \end{pmatrix}$.

$$\begin{pmatrix} 1-\alpha & 1 \\ 1 & -\alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\left(\begin{array}{cc|c} 1-\alpha & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} \beta & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & 1/\beta & 0 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -\alpha & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Solution $x - \alpha y = 0$

$$x = \alpha y.$$

This is the line $\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} \alpha \\ 1 \end{pmatrix}$.

$\lambda = \beta$: $$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \beta \begin{pmatrix} x \\ y \end{pmatrix}.$$

$$\begin{pmatrix} 1-\beta & 1 \\ 1 & -\beta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{cc|c} 1-\beta & 1 & 0 \\ 1 & -\beta & 0 \end{array}\right. \longrightarrow \left.\begin{array}{cc|c} 1-\beta & 1 & 0 \\ 0 & 0 & 0 \end{array}\right..$$

$$\longrightarrow \left.\begin{array}{cc|c} \alpha & 1 & 0 \\ 0 & 0 & 0 \end{array}\right. \longrightarrow \left.\begin{array}{cc|c} 1 & 1/\alpha & 0 \\ 0 & 0 & 0 \end{array}\right.$$

$$\longrightarrow \left.\begin{array}{cc|c} 1 & -\beta & 0 \\ 0 & 0 & 0 \end{array}\right. \longrightarrow \begin{array}{l} x - \beta y = 0 \\ x = \beta y. \end{array}$$

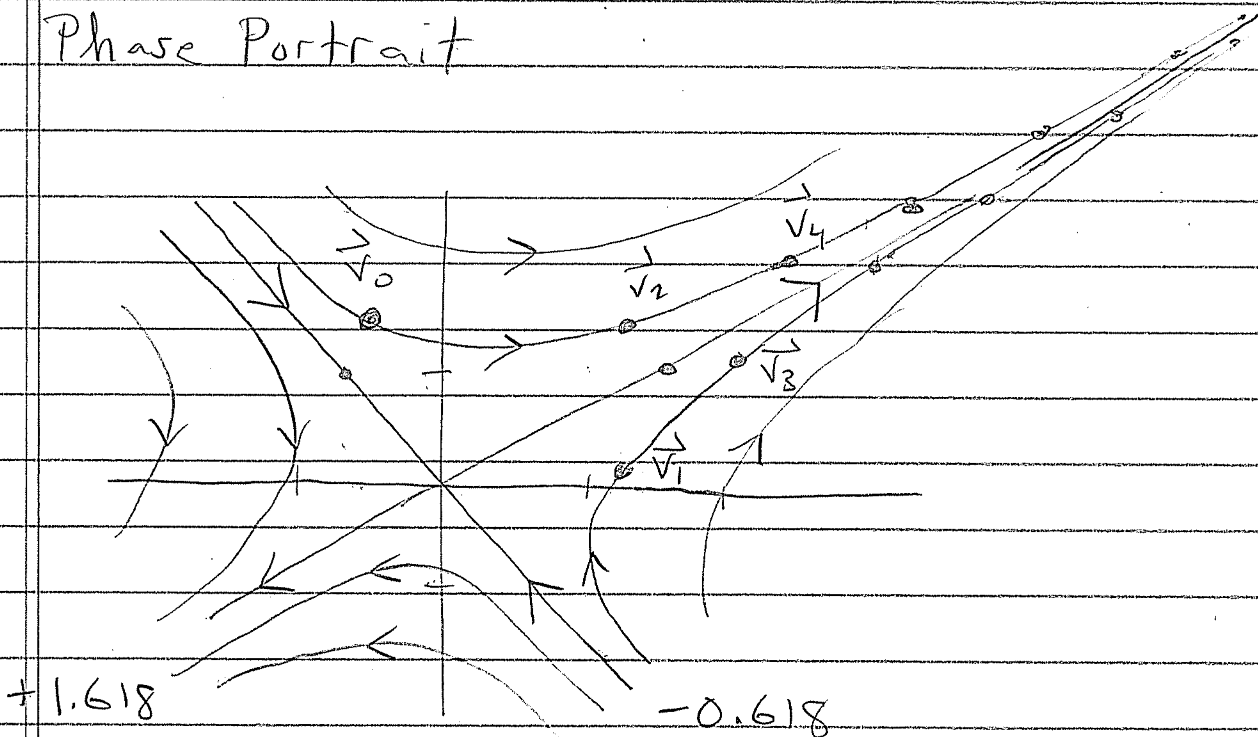
This is the line $\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} \beta \\ 1 \end{pmatrix}$.

Now observe

$\alpha = \dfrac{1+\sqrt{5}}{2} \approx 1.618$ "the golden ratio"

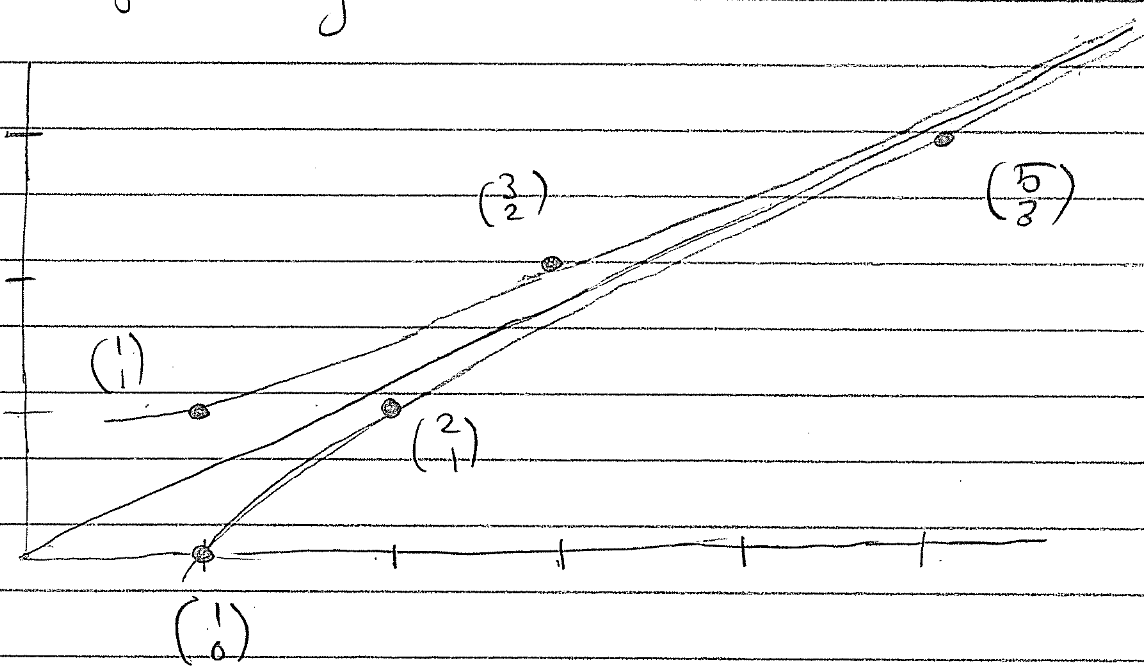
$\beta = \dfrac{1-\sqrt{5}}{2} \approx -0.618$

Phase Portrait



<u>But</u> the −0.618 hops back and forth.

Our Trajectory.



The points $\vec{v}_n = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$ get very close to the line $t\begin{pmatrix} \alpha \\ 1 \end{pmatrix}$. In other words,

$$\frac{F_{n+1}}{F_n} \approx \frac{\alpha}{1} \approx 1.618 \quad \text{"golden ratio"}$$

This means that

$$F_n \approx C\,\alpha^n = C\,(1.618)^n$$

for some constant C
what is the constant ?

Step 1: Express $\vec{v}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in terms of eigenvectors:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} \alpha \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}}\begin{pmatrix} \beta \\ 1 \end{pmatrix}$$

Details omitted.

Step 2: Apply A^n.

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$= A^n \left[\frac{1}{\sqrt{5}} \begin{pmatrix} \alpha \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}} \begin{pmatrix} \beta \\ 1 \end{pmatrix} \right]$$

$$= \frac{1}{\sqrt{5}} A^n \begin{pmatrix} \alpha \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}} A^n \begin{pmatrix} \beta \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \alpha^n \begin{pmatrix} \alpha \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}} \beta^n \begin{pmatrix} \beta \\ 1 \end{pmatrix}$$

$$\Longrightarrow F_n = \frac{1}{\sqrt{5}} \alpha^n - \frac{1}{\sqrt{5}} \beta^n.$$

$$= \frac{1}{\sqrt{5}} (1.618)^n - \frac{1}{\sqrt{5}} (-0.618)^n$$

$$\approx \frac{1}{\sqrt{5}} (1.618)^n$$

Because $\frac{1}{\sqrt{5}}(-0.618)^n \longrightarrow 0$

as $n \longrightarrow \infty$.

Wed Apr 17

Exam 2 on Friday

Today: Review

What did we do?

1) Projection / Least Squares

2) Eigenvalues / vectors.

1) Start with $A\vec{x} = \vec{b}$.

Suppose it has no solution $\vec{x}$, i.e.

$\vec{b} \neq A(\text{something})$.

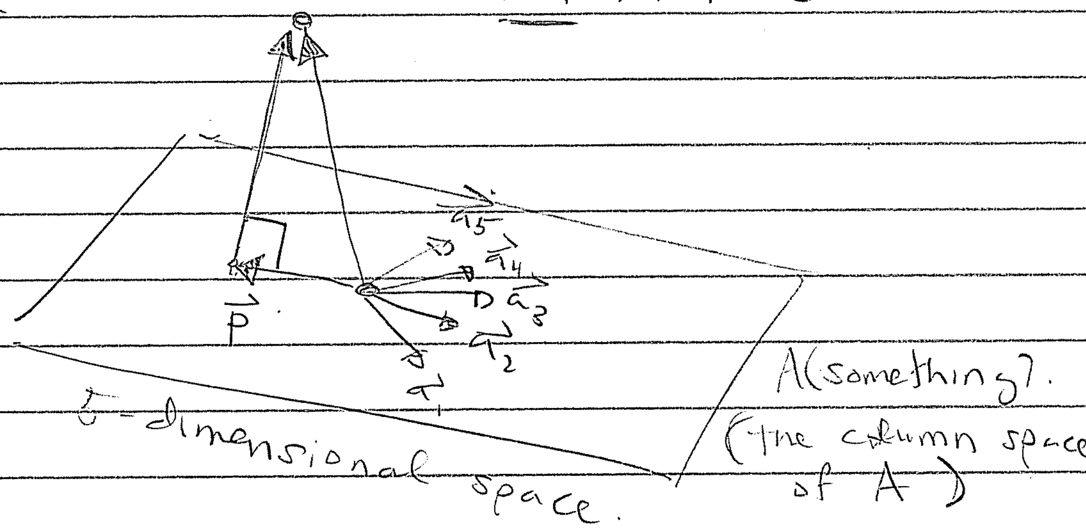
What does $A(\text{something})$ look like.

If $A = (\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_5)$ (5 columns).

Then

$$A\vec{x} = A\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{pmatrix} = \underbrace{x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_5\vec{a}_5}_{\text{linear combination of the columns.}}$$

Picture: $\vec{b}$ is not in there



Find the point $\vec{p}$ in the space, closest to $\vec{b}$.

(i) Since $\vec{p}$ is in the space we have

$$\vec{p} = A(\text{something}) = A\hat{x}.$$

Goal: Find $\hat{x}$.

(ii) Since $\vec{p}$ is closest to $\vec{b}$, the error $\vec{e} = \vec{b} - \vec{p} = \vec{b} - A\hat{x}$ is orthogonal to the space.

i.e. $\vec{e}$ is orthogonal to each of $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_5$

i.e. $\vec{a}_1^T \vec{e} = 0$

$$\vec{a}_2^T \vec{e} = 0$$

$$\vdots$$

$$\vec{a}_5^T \vec{e} = 0.$$

i.e. $A^T \vec{e} = \vec{0}$.

Put (i) and (ii) together

$$A^T \vec{e} = \vec{0}$$
$$A^T(\vec{b} - A\hat{x}) = \vec{0}$$
$$A^T\vec{b} - A^T A\hat{x} = \vec{0}$$
$$A^T\vec{b} = A^T A \hat{x}.$$

Now we can solve for $\hat{x}$:

Since (A^TA) is invertible

$$\Longrightarrow \hat{x} = (A^TA)^{-1}A^T\vec{b}.$$

Now we can solve for the projection $\vec{p}$.

$$\vec{p} = A\hat{x} = \underbrace{A(A^TA)^{-1}A^T}_{\text{the projection matrix}}\vec{b}.$$

Easiest Case: Project onto a line $t\vec{a}$.

Then $A = \vec{a}$ is just a column vector.

The projection matrix is

$$P = \vec{a}(\vec{a}^T\vec{a})^{-1}\vec{a}^T$$

But $\vec{a}^T\vec{a}$ is a 1×1 matrix (i.e. a number),

so $(\vec{a}^T\vec{a})^{-1} = \frac{1}{\vec{a}^T\vec{a}}$. (Easy).

Hence

$$P = \vec{a}(\vec{a}^T\vec{a})^{-1}\vec{a}^T = \underbrace{\frac{1}{\vec{a}^T\vec{a}}}_{\text{number}}\underbrace{(\vec{a}\vec{a}^T)}_{\text{matrix}}.$$

(2) Eigen.

Let A be square. We say $\vec{x}$ is an eigenvector if

- $\vec{x} \neq \vec{0}$
- $A\vec{x} = \lambda\vec{x}$ for some number λ (the eigenvalue).

Q: Which numbers λ could be eigenvalues?

We need
$$A\vec{x} = \lambda\vec{x}$$
$$A\vec{x} = \lambda I\vec{x}$$
$$A\vec{x} - \lambda I\vec{x} = \vec{0}.$$
$$(A - \lambda I)\vec{x} = \vec{0}.$$

for some $\vec{x} \neq \vec{0}$. In other words, the matrix $A - \lambda I$ must be singular (i.e. non-invertible).

For 2x2 matrices this is easy to say:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

is singular $\Leftrightarrow$ the rows are parallel

$$\Leftrightarrow \quad \frac{a-\lambda}{b} = \frac{c}{d-\lambda}$$

$$\Leftrightarrow \quad \underbrace{(a-\lambda)(d-\lambda) - bc = 0}_{\substack{\text{The characteristic equation.} \\ \text{The solutions are the eigenvalues}}}.$$

[Remark: Another language says

$$\det(A - \lambda I) = 0.$$]

Once you get the eigenvalues, the eigenvectors are easy:

Just solve $(A - \lambda I)\vec{x} = \vec{0}$ using your favorite method.

Q: Who Cares?

If you want to solve a linear recurrence

$$\vec{V}_{n+1} = A\vec{V}_n,$$

(i) Find the eigenvectors $A\vec{x}_1 = \lambda_1 \vec{x}_1$
$A\vec{x}_2 = \lambda_2 \vec{x}_2$

(ii) Express $\vec{V}_0 = s\vec{x}_1 + t\vec{x}_2$.

(iii) The solution is $\vec{V}_n = A^n \vec{V}_0$

$$= sA^n\vec{x}_1 + tA^n\vec{x}_2 = s(\lambda_1)^n\vec{x}_1 + t(\lambda_2)^n\vec{x}_2.$$

Enjoy!