Wed Mar 27.

HW 7 EXTENDED to Mon Apr 1.

Today: Least Squares.

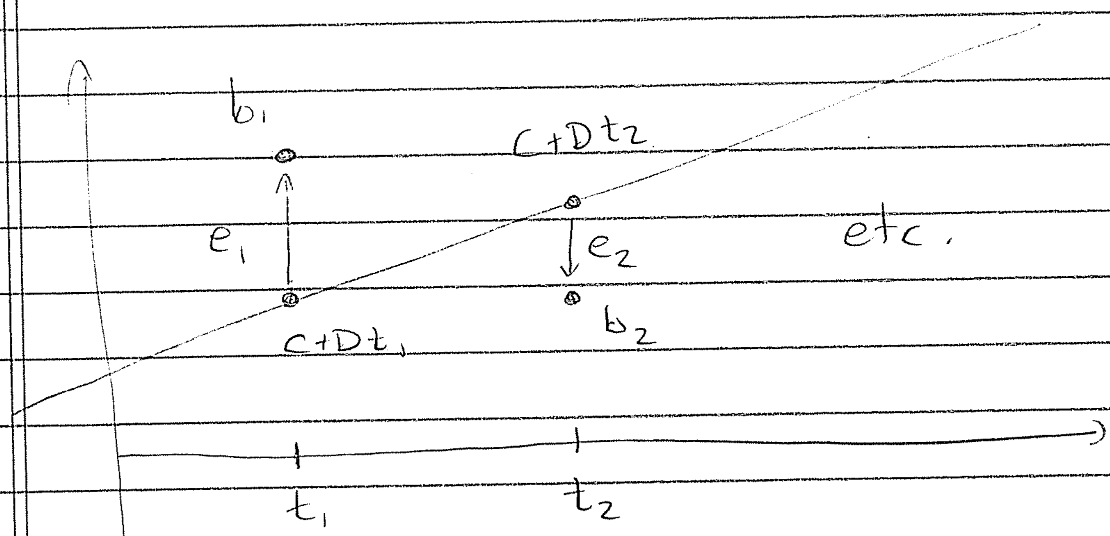
(1) Best Fit Line with Linear Algebra.

Given points $\begin{pmatrix} t_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} t_2 \\ b_2 \end{pmatrix}, \dots, \begin{pmatrix} t_m \\ b_m \end{pmatrix}$,

find the line $C + Dt = b$ such that the error vector

$$\vec{e} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{pmatrix} = \begin{pmatrix} b_1 - (C + Dt_1) \\ b_2 - (C + Dt_2) \\ \vdots \\ b_m - (C + Dt_m) \end{pmatrix}$$

has <u>minimum</u> length



i.e. minimize $e_1^2 + e_2^2 + \cdots + e_m^2$.

Answer: Let $A = \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{pmatrix}$

Then $\|\vec{e}\|^2$ (and hence $\|\vec{e}\|$) is minimized for

$$A^T A \begin{pmatrix} C \\ D \end{pmatrix} = A^T \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

$$\Longrightarrow \begin{pmatrix} m & \Sigma t_i \\ \Sigma t_i & \Sigma t_i^2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} \Sigma b_i \\ \Sigma t_i b_i \end{pmatrix}.$$

Solve for C & D.

(2) Best Fit Line with Calculus.

Find C & D that minimize

$$e_1^2 + e_2^2 + \cdots + e_m^2$$

$$= (b_1 - C - Dt_1)^2 + \cdots + (b_m - C - Dt_m)^2.$$

$$= b_1^2 - 2b_1(C+Dt_1) + (C+Dt_1)^2 + \cdots + b_m^2 - 2b_m(C+Dt_m) + (C+Dt_m)^2.$$

$$\left.\begin{aligned} = \; & b_1^2 - 2b_1C - 2b_1Dt_1 + C^2 + 2CDt_1 + D^2t_1^2 \\ & + \cdots \\ & \;\vdots \\ & + b_m^2 - 2b_mC - 2b_mDt_m + C^2 + 2CDt_m + D^2t_m^2 \end{aligned}\right\} \; (*)$$

How? Set derivatives of $(*)$ to zero.

Let $\frac{d}{dC}(*) = 0$:

$$\begin{aligned} & -2b_1 + 2C + 2Dt_1 \\ & + \cdots \\ & \;\vdots \\ & + -2b_m + 2C + 2Dt_m \quad = 0. \end{aligned}$$

Divide by 2 to get.

$$Cm + D\sum t_i = \sum b_i.$$

Next let $\frac{d}{dD}(*) = 0$:

$$\begin{aligned} & -2b_1t_1 + 2Ct_1 + 2Dt_1^2 \\ & + \cdots \\ & \;\vdots \\ & + -2b_mt_m + 2Ct_m + 2Dt_m^2 \quad = 0. \end{aligned}$$

Divide by 2 to get

$$C \sum t_i + D \sum t_i^2 = \sum t_i b_i$$

Hence $\|\vec{e}\|^2$ (and hence $\|\vec{e}\|$) is minimized when

$$\begin{cases} Cm + D \sum t_i = \sum b_i \\ C \sum t_i + D \sum t_i^2 = \sum t_i b_i \end{cases}$$

SAME ANSWER ☺

But the linear algebra is <u>better</u> because it is easier to <u>generalize</u>.

For example, to compute the elliptical orbit of an asteroid, given some data points (Gauss and Ceres, 1801).

The general method: If you naively want to solve $A\vec{x} = \vec{b}$ and it has no solution,

try $A^T A \hat{x} = A^T \vec{b}$ instead!

Example: Given the points

$$\begin{pmatrix} t \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Find the best fit parabola of the form

$$C + Dt + Et^2 = b.$$

Answer: Try to solve

$$\begin{aligned} C + 0 + 0 &= 3 \\ C + D + E &= 1 \\ C + 2D + 4E &= 0 \\ C + 3D + 9E &= 1. \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$A \vec{x} = \vec{b}.$$

NO SOLUTION.

So instead solve $A^T A \hat{x} = A^T \vec{b}$

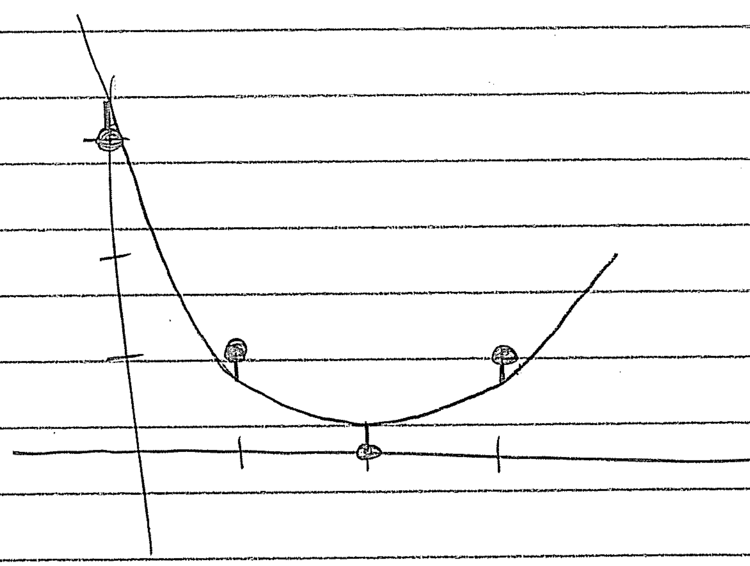
$$\begin{pmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{pmatrix} \begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 10 \end{pmatrix}.$$

$\Big\}$ some work

$$\begin{pmatrix} C \\ D \\ E \end{pmatrix} = \frac{1}{20} \begin{pmatrix} 61 \\ -59 \\ 15 \end{pmatrix}$$

Best fit parabola: $\frac{61}{20} - \frac{59}{20} t + \frac{15}{20} t^2 = b$.

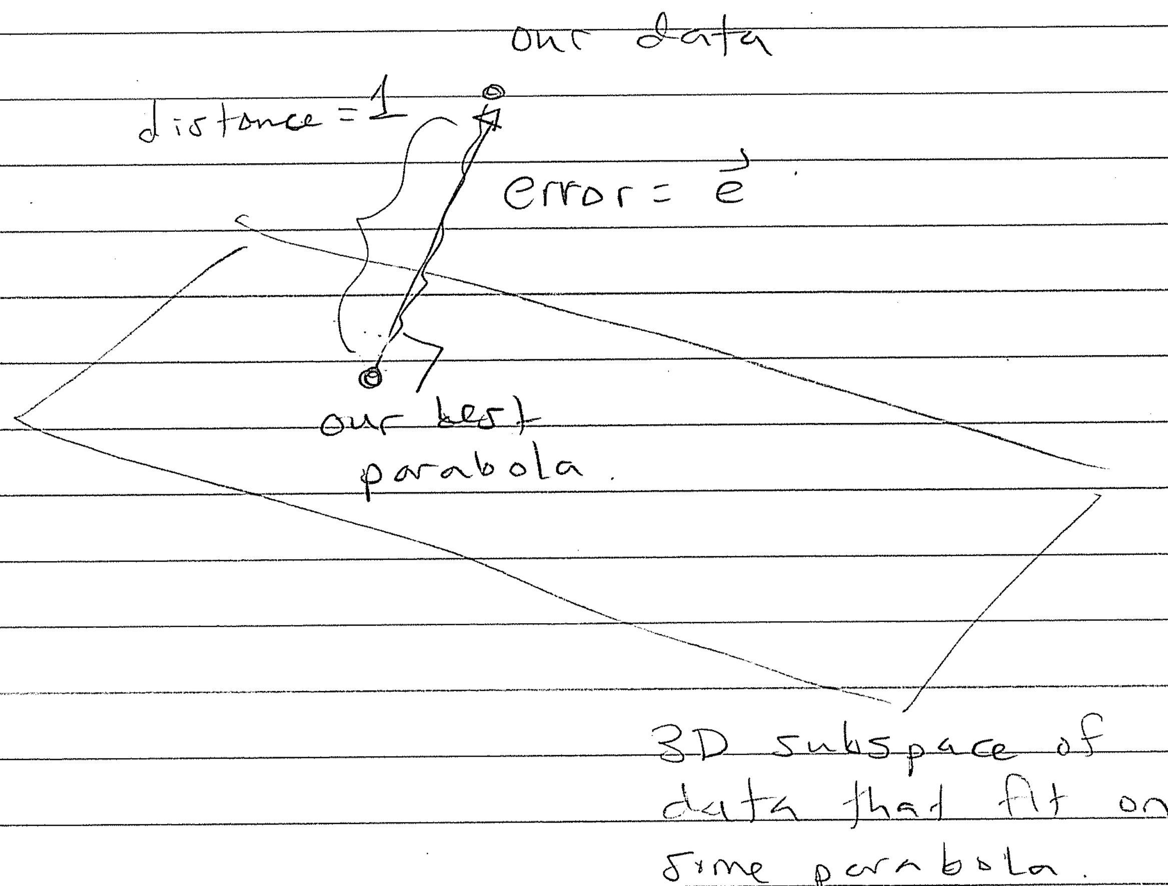
Picture.



The error is $\vec{e} = \frac{1}{20}\begin{pmatrix} -1 \\ 3 \\ -3 \\ 1 \end{pmatrix}$

So $\|\vec{e}\|^2 = \frac{1+9+9+1}{20} = 1$

Picture in 4D "Data Space"



Fri Mar 29.

HW 7 Extended until Monday

Hints for the Homework:

Suppose you want to find the parabola $C + Dt + Et^2 = b$ closest to points

$$\begin{pmatrix} t \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

But you insist that the line must contain $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$. What do you do?

You are requiring that

$$C + D0 + E0^2 = 3$$
$$C = 3.$$

So you really want a parabola of the form $3 + Dt + Et^2 = b$

$$Dt + Et^2 = b - 3$$

closest to the points

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Set it up:

$$\begin{aligned} D + E &= -2 \\ 2D + 4E &= -3 \\ 3D + 9E &= -2 \end{aligned} \longrightarrow \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \end{pmatrix} \begin{pmatrix} D \\ E \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ -2 \end{pmatrix}$$

$$A \vec{x} = \vec{b}$$

NO SOLUTION, so we solve the normal equation

$$A^T A \hat{x} = A^T \vec{b}$$

$$\begin{pmatrix} 14 & 36 \\ 36 & 98 \end{pmatrix} \begin{pmatrix} D \\ E \end{pmatrix} = \begin{pmatrix} -11 \\ -23 \end{pmatrix}$$

some work

$$D = \frac{-125}{38}, \qquad E = \frac{37}{38}$$

Your parabola is

$$3 - \frac{125}{38} t + \frac{37}{38} t^2 = b$$

The method $A\vec{x} = \vec{b} \longrightarrow A^T A \hat{x} = A^T \vec{b}$
(invented by Gauss) is very powerful.

Example: Find the line of the form
$C = b$ (horizontal line) that
best fits the points

$$\begin{pmatrix} t_1 \\ b_1 \end{pmatrix}, \ldots, \begin{pmatrix} t_m \\ b_m \end{pmatrix}$$

Set it up. Write the naive equation

$$\begin{aligned} C &= b_1 \\ C &= b_2 \\ &\vdots \\ C &= b_m. \end{aligned} \longrightarrow \underbrace{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}}_{A} \underbrace{(C)}_{\vec{x}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{\vec{b}}$$

Obviously this has no solution.

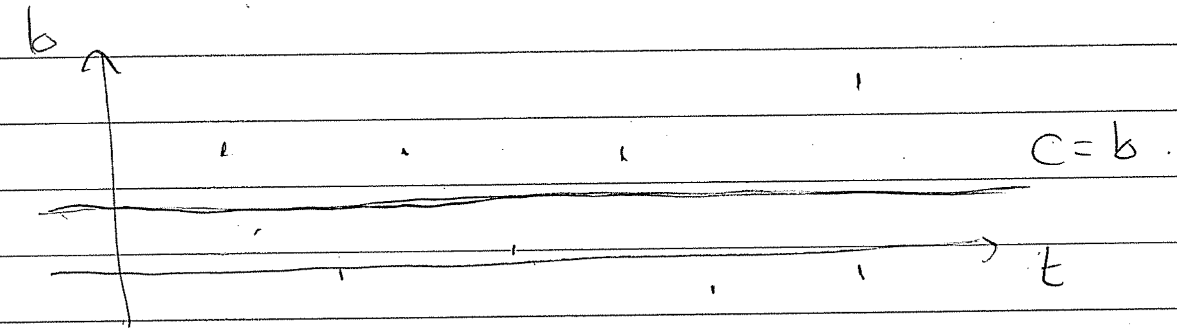
$A^T A \hat{x} = A^T \vec{b}$

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} (C) = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$mC = b_1 + b_2 + \cdots + b_m$$

$$C = \frac{b_1 + b_2 + \cdots + b_m}{m}$$

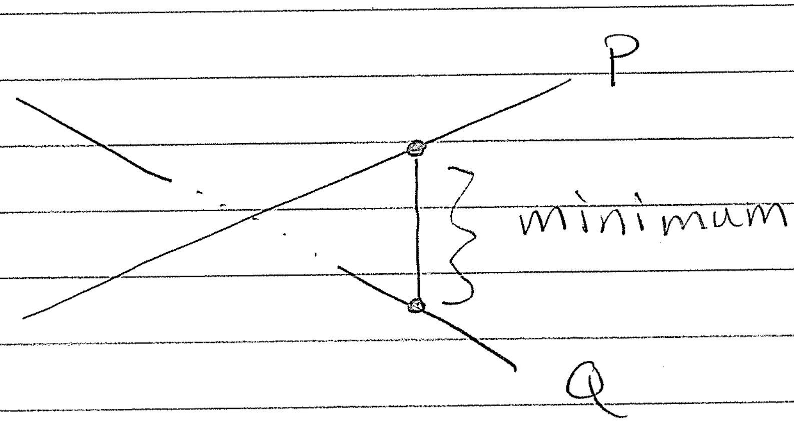
Just the average value ☺



Example: Problem 4.3.27

Choose x and y to minimize the (squared) distance between the lines

$$P = \begin{pmatrix} x \\ x \\ x \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} y \\ 3y \\ -1 \end{pmatrix}$$



3 solutions.

1 Calculus.

$$F = (\text{distance})^2 = \left\| \begin{pmatrix} x \\ x \\ x \end{pmatrix} - \begin{pmatrix} y \\ 3y \\ -1 \end{pmatrix} \right\|^2$$

$$= (x-y)^2 + (x-3y)^2 + (x+1)^2$$

Set $\frac{dF}{dx} = 0$

$$2(x-y) + 2(x-3y) + 2(x+1) = 0$$
$$6x - 8y = -2$$
$$3x - 4y = -1.$$

Set $\frac{dF}{dy} = 0$

$$-2(x-y) - 3 \cdot 2(x-3y) = 0$$
$$-8x + 20y = 0.$$
$$4x - 10y = 0.$$

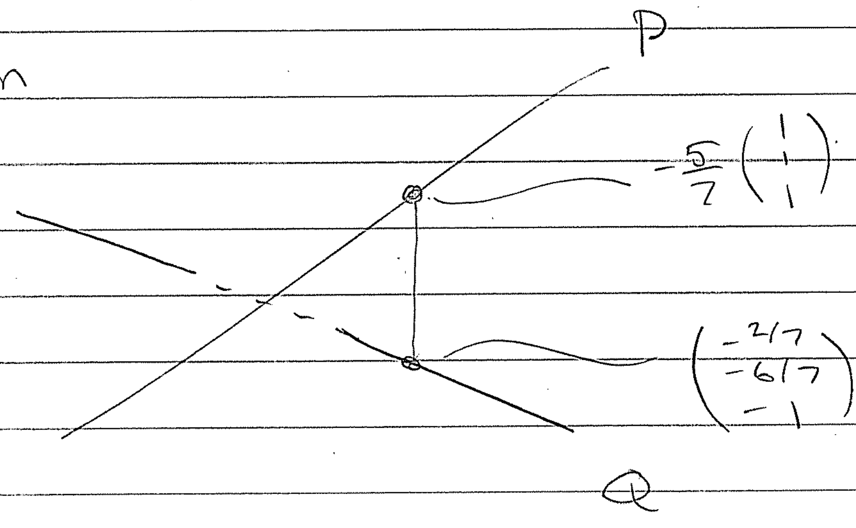
Solve $\begin{cases} 3x - 4y = -1 \\ 4x - 10y = 0. \end{cases}$

$$\left.\begin{array}{cc|c} 3 & -4 & -1 \\ 4 & -10 & 0 \end{array}\right. \to \left.\begin{array}{cc|c} 3 & -4 & -1 \\ 0 & -14/3 & 4/3 \end{array}\right.$$

$$\to \left.\begin{array}{cc|c} 3 & -4 & -1 \\ 0 & 1 & -2/7 \end{array}\right. \to \left.\begin{array}{cc|c} 3 & 0 & -15/7 \\ 0 & 1 & -2/7 \end{array}\right.$$

$$\to \left.\begin{array}{cc|c} 1 & 0 & -5/7 \\ 0 & 1 & -2/7 \end{array}\right.$$

Solution

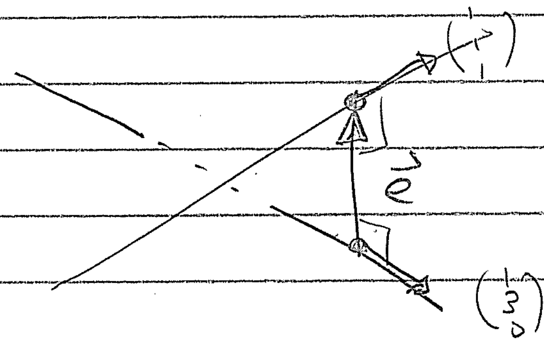


(2) Geometry.

The error vector $\vec{e} = \begin{pmatrix} x \\ x \\ x \end{pmatrix} - \begin{pmatrix} y \\ 3y \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} x - y \\ x + 3y \\ x + 1 \end{pmatrix}$$

should be perpendicular to both lines



$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\vec{e} = 0 \quad \& \quad \begin{pmatrix}1 & 3 & 0\end{pmatrix}\vec{e} = 0$$

$$(x-y)+(x-3y)+(x+1) = 0 \quad \& \quad (x-y)+3(x-3y) = 0$$

$$3x-4y = -1 \quad \& \quad 4x-10y = 0$$

SAME AS BEFORE,
but easier.

Easier still is ...

(3) The normal equation:

Be naive. Try to intersect the lines,

Find x, y such that

$$\begin{pmatrix} x \\ x \\ x \end{pmatrix} = \begin{pmatrix} y \\ 3y \\ -1 \end{pmatrix}$$

$$x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 1 & -3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\text{"} A \vec{x} = \vec{b} \text{"}$$

No solution so write.

$$A^T A \hat{x} = A^T \vec{b}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & -3 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & -3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -4 \\ -4 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

No thinking involved

Mon Apr 1

HW 7 due NOW
HW 8 due next Monday

New Topic : Eigenthings (Google)
(Chapter 6)

Introduction : How could we solve
the following equation

$$f(n) = f(n-1) + f(n-2)$$

Find $f(n)$ for all whole numbers n.
NOT ENOUGH INFORMATION.
We need "boundary conditions", so
let's say

$$f(0) = 0 \quad \text{and} \quad f(1) = 1.$$

Then $f(2) = f(1) + f(0) = 1 + 0 = 1$
$f(3) = f(2) + f(1) = 1 + 1 = 2$
$f(4) = f(3) + f(2) = 2 + 1 = 3$

it continues...

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...
"The Fibonacci sequence"

But I want a <u>formula</u> for $f(n)$.

Use Linear Algebra!

For each n, define the vector

$$\vec{f}_n = \begin{pmatrix} f(n+1) \\ f(n) \end{pmatrix}$$

In this language, the recurrence equation is

$$\vec{f}_{n+1} = \begin{pmatrix} f(n+2) \\ f(n+1) \end{pmatrix} = \begin{pmatrix} f(n+1) + f(n) \\ f(n+1) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f(n+1) \\ f(n) \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{Call this "transition matrix" } T} \vec{f}_n$$

Initial condition:

$$\vec{f}_0 = \begin{pmatrix} f(1) \\ f(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Recurrence:

$$\vec{f}_{n+1} = T\vec{f}_n \quad \text{for all } n \geq 0$$

It follows that

$$\vec{f}_1 = T\vec{f}_0$$

$$\vec{f}_2 = T\vec{f}_1 = TT\vec{f}_0 = T^2\vec{f}_0$$

$$\vec{f}_3 = T\vec{f}_2 = TTT\vec{f}_0 = T^3\vec{f}_0$$

and in general

$$\boxed{\vec{f}_n = T^n\vec{f}_0 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

So the n^{th} fibonacci number is the 2nd coordinate in

$$\begin{pmatrix} f(n+1) \\ f(n) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Compute it! How?

There are certain initial conditions for which this is easy.

Suppose we had a vector $\vec{x}$ such that $T\vec{x} = \lambda \vec{x}$ for some number λ.

Notation: Then $\vec{x}$ is called an eigenvector for T with eigen<u>value</u> λ

In this case

$$T^n \vec{x} = \lambda^n \vec{x}. \quad \text{Easy.}$$

Unfortunately, $\binom{1}{0}$ is <u>not</u> an eigenvector for T.

<u>Big Idea</u>: What if we could express $\binom{1}{0}$ as a linear combination of eigenvectors...

Say $\binom{1}{0} = A\vec{u} + B\vec{v}$,

where $T\vec{u} = \lambda \vec{u}$ and $T\vec{v} = \mu \vec{v}$.

Then $T^n\begin{pmatrix}1\\0\end{pmatrix} = T^n(A\vec{u} + B\vec{v})$

$$= AT^n\vec{u} + BT^n\vec{v}$$

$$= A\lambda^n\vec{u} + B\mu^n\vec{v}$$

and we're done!

We must compute the eigenvectors of T.
For now I'll just <u>tell</u> you
(believe me) that

$$\left.\begin{aligned} & T\begin{pmatrix}\frac{1+\sqrt{5}}{2}\\1\end{pmatrix} = \left(\frac{1+\sqrt{5}}{2}\right)\begin{pmatrix}\frac{1+\sqrt{5}}{2}\\1\end{pmatrix} \\ \& \quad & T\begin{pmatrix}\frac{1-\sqrt{5}}{2}\\1\end{pmatrix} = \left(\frac{1-\sqrt{5}}{2}\right)\begin{pmatrix}\frac{1-\sqrt{5}}{2}\\1\end{pmatrix}\end{aligned}\right\} \text{These are the eigenvectors}$$

Then I'll just tell you that

$$\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}\frac{1+\sqrt{5}}{2}\\1\end{pmatrix} - \frac{1}{\sqrt{5}}\begin{pmatrix}\frac{1-\sqrt{5}}{2}\\1\end{pmatrix}$$

in terms of eigenvectors.

[Computations are easy, but we'll do them later.]

Finally, this tells us that

$$T^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = T^n \left(\frac{1}{\sqrt{5}} \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}} \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix} \right)$$

$$= \frac{1}{\sqrt{5}} T^n \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}} T^n \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} .$$

Since $T^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} f(n+1) \\ f(n) \end{pmatrix}$, we conclude

$$f(n) = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$

This is the nth Fibonacci number.

Amazing!

It doesn't even look like a whole number!

Wed April 3

HW 8 due MONDAY.
Office Hours 3-4 Today

Last time I tried to motivate
eigenvalues and eigenvectors.

Now we'll compute them.
This is Section 6.1.

Recall: Given a square matrix A,
if we have vector $\vec{x}$ and number λ with

$$\boxed{A\vec{x} = \lambda\vec{x}}$$

then we say $\vec{x}$ is an eigenvector
for A with eigenvalue λ.
(i.e. The function A doesn't change
the direction of $\vec{x}$.)

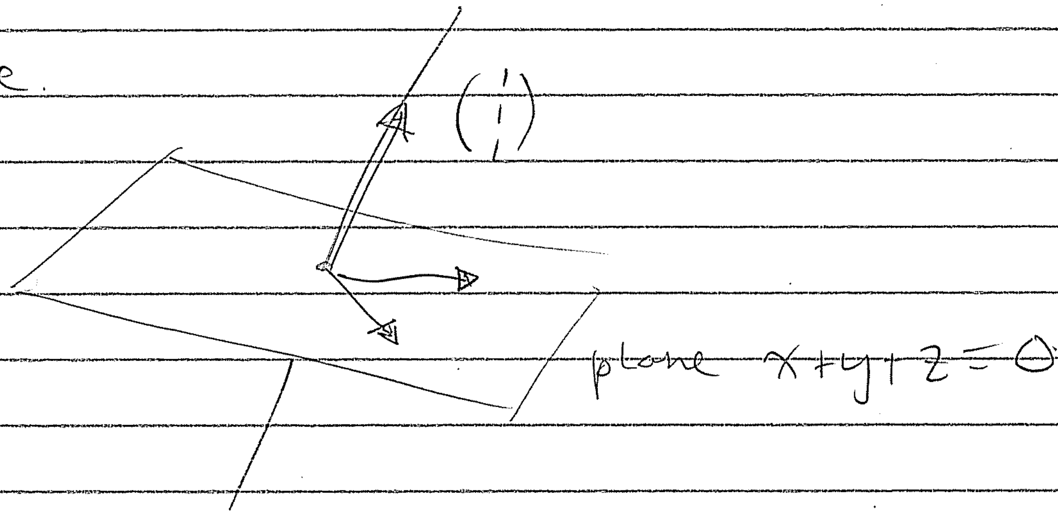
Example: The matrix

$$P = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

projects onto the line through $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

What are its eigenvectors / eigenvalues?

Picture.



Any vector on the line, say $\begin{pmatrix}t\\t\\t\end{pmatrix}$ projects to itself.

$$P\begin{pmatrix}t\\t\\t\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}t\\t\\t\end{pmatrix} = \frac{1}{3}\begin{pmatrix}t+t+t\\t+t+t\\t+t+t\end{pmatrix} = \begin{pmatrix}t\\t\\t\end{pmatrix}.$$

So $\begin{pmatrix}t\\t\\t\end{pmatrix}$ is an e.vector with e.value 1

In fact we have a whole space of e.vectors with e.value 1 — the line. This is called the <u>eigen</u>space of <u>eigenvalue</u> 1.

Any more e.vectors / e.values?

The vectors in the plane are

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y-z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

What if we project these?

$$P\left(y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right) = y\,P\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\,P\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$= y\,\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\,\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$= y\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= 0\left(y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right)$$

Any vector in the plane is an e.vector with e.value 0.

The plane is the 0-eigenspace.

Any more? NO.

P changes the direction of every other vector.

You see the e.vectors tell us a lot about a matrix.
How to compute them if we don't already know them

Problem: Solve

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

Clearly there is a trivial solution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for any } \lambda.$$

But that doesn't count!

$$ax + by = \lambda x$$
$$cx + dy = \lambda y.$$

$$\rightarrow \quad (a-\lambda)x + by = 0$$
$$cx + (d-\lambda)y = 0.$$

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We require at least a line of solutions so we must have

$$\begin{pmatrix} a-\lambda \\ c \end{pmatrix} = K \begin{pmatrix} b \\ d-\lambda \end{pmatrix} \quad \text{for some } K.$$

$$(a-\lambda) = Kb \quad \rightarrow \quad K = (a-\lambda)/b$$
$$c = K(d-\lambda) \qquad K = c/(d-\lambda)$$

Hence

$$(a-\lambda)/b = c/(d-\lambda).$$

$$\Rightarrow (a-\lambda)(d-\lambda) = bc.$$

$$\Rightarrow \underbrace{(a-\lambda)(d-\lambda) - bc}_{\text{This is called the "characteristic polynomial"}} = 0.$$

The roots λ are precisely the eigenvalues.

$$(a-\lambda)(d-\lambda) - bc = 0$$
$$ad - a\lambda - d\lambda + \lambda^2 - bc = 0$$

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

$$\lambda = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

<u>Example</u>: Find the eigenvalues of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Characteristic Equation

$$(1-\lambda)(0-\lambda) - 1 \cdot 1 = 0$$
$$-\lambda + \lambda^2 - 1 = 0$$
$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{(-1)^2 - 4(-1)}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2} \quad \text{Recognize?}$$

Once we know the eigenvalues, the corresponding eigenvectors are easy to compute

Just solve the system $A\vec{x} = \lambda \vec{x}$ for $\vec{x}$, given your value of λ.

Example: Let $\varphi = (1+\sqrt{5})/2$ and note that

$$\varphi^2 - \varphi - 1 = 0 \text{ is true.}$$

Then solve $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \varphi \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} 1-\varphi & 1 \\ 1 & 0-\varphi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1-\varphi & 1 & 0 \\ 1 & -\varphi & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1-\varphi & 1 & 0 \\ 0 & -\varphi - \frac{1}{1-\varphi} & 0 \end{array}\right)$$

Note that $-\varphi - \dfrac{1}{1-\varphi} = \dfrac{-\varphi(1-\varphi) - 1}{1-\varphi}$

$$= \frac{\varphi^2 - \varphi - 1}{1-\varphi} = 0.$$

So we get
$$\left(\begin{array}{cc|c} 1-4 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

which is good :)

Let $x=t$ be a parameter, then we get solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ t(1-4) \end{pmatrix} = t\begin{pmatrix} 1 \\ 1-4 \end{pmatrix}.$$

This line is the "4-eigenspace" of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$