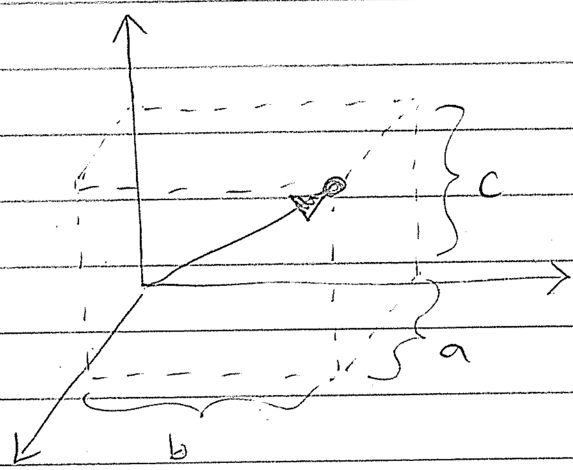
Wed Feb 27

Exam Friday
- no cheating.

Today: Review.

Cartesian coordinates (1637):



The point/vector is called $\vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Vectors can be added and scalar multiplied ("linear combinations").

Given $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$,

The set of linear combinations

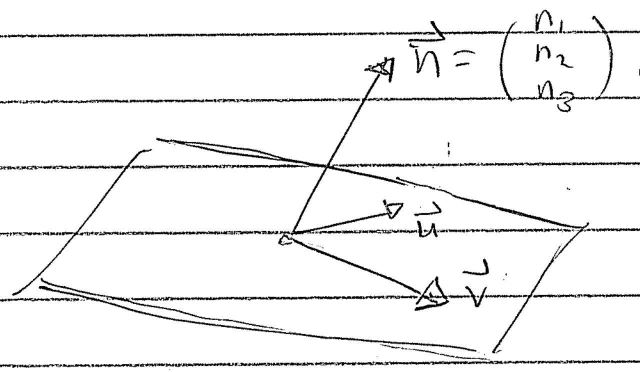
$$s\vec{u} + t\vec{v} = s\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} s+t \\ s+2t \\ s+3t \end{pmatrix}$$

form a plane in $\mathbb{R}^3$

s, t are called parameters or free variables.

We can also express this plane with a single equation. How?

Find a normal vector to the plane.



We want $\vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ such that

$$\begin{cases} \vec{n} \cdot \vec{u} = 0 \\ \vec{n} \cdot \vec{v} = 0 \end{cases} \quad \text{i.e.} \quad \begin{cases} n_1 + n_2 + n_3 = 0 & (1) \\ n_1 + 2n_2 + 3n_3 = 0. & (2) \end{cases}$$

Do Gaussian Elimination.

$$\begin{cases} n_1 + n_2 + n_3 = 0 & (1)' = (1) \\ 0 + n_2 + 2n_3 = 0. & (2)' = (2) - (1) \end{cases}$$

$$\begin{cases} n_1 + 0 - n_3 = 0 & (1)'' = (1)' - (2)' \\ 0 + n_2 + 2n_3 = 0 & (2)'' = (2)'. \end{cases}$$

Reduced Row Echelon Form ✓

Interpret :

Pivot variables n_1, n_2
Free parameters n_3.

Let $n_3 = r$. Then

$$n_1 + 0 - r = 0 \implies n_1 = r$$
$$0 + n_2 + 2r = 0 \qquad n_2 = -2r.$$

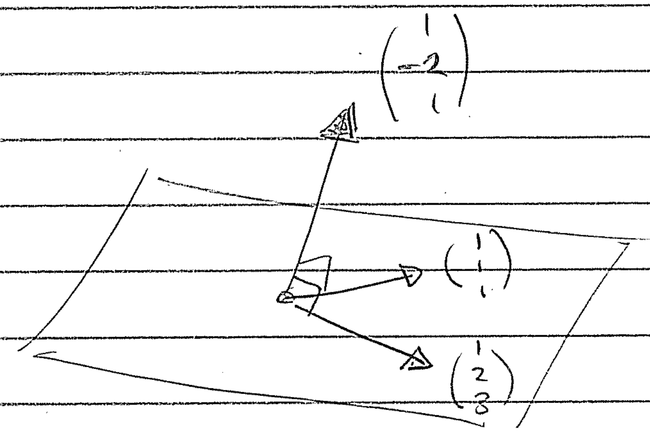
Hence

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} r \\ -2r \\ r \end{pmatrix} = r \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$ a line in $\mathbb{R}^3$.

We could also have used the cross product

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3-2 \\ 1-3 \\ 2-1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

So we have



The line can be expressed as

$$r\begin{pmatrix}1\\-2\\1\end{pmatrix}$$ parametrized

OR
$$\begin{cases} x+y+z=0 \\ x+2y+3z=0 \end{cases}$$ implicit.

The plane can be expressed as

$$s\begin{pmatrix}1\\1\\1\end{pmatrix}+t\begin{pmatrix}1\\2\\3\end{pmatrix}$$ parametrized

OR
$$\boxed{x-2y+z=0}$$ implicit.

$$\underbrace{(x\ y\ z)\begin{pmatrix}1\\-2\\1\end{pmatrix}=0}_{\text{all vectors }\perp\text{ to }\begin{pmatrix}1\\-2\\1\end{pmatrix}}$$

The central problem of linear algebra is to solve a system of m linear equations in n unknowns.

Typically the solution has

$n-m$ free parameters if $m \leq n$.

If $m > n$ there is typically NO SOLUTION.

Geometrically: We compute the intersection of m hyperplanes in $\mathbb{R}^n$.

How do we do it? Gaussian Elimination.

Intersect 3 hyperplanes in $\mathbb{R}^5$

$$\begin{cases} x_1 - x_2 + x_3 + 3x_4 + 0 = 1 \\ x_1 - x_2 - x_3 + x_4 + x_5 = 2 \\ 2x_1 - 2x_2 + x_3 + 5x_4 + x_5 = 1 \end{cases}$$

$$\Downarrow$$

$$\begin{aligned} x_1 - x_2 + 0 + 2x_4 + 0 &= 3 \\ x_3 + x_4 + 0 &= -2 \\ x_5 &= -3 \end{aligned}$$

RREF

x_2, x_4 are free.

Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 + x_2 - 2x_4 \\ x_2 \\ -2 - x_4 \\ x_4 \\ -3 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \\ -3 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}}_{\text{solutions form a 2D plane in } \mathbb{R}^5}$$

Finally:

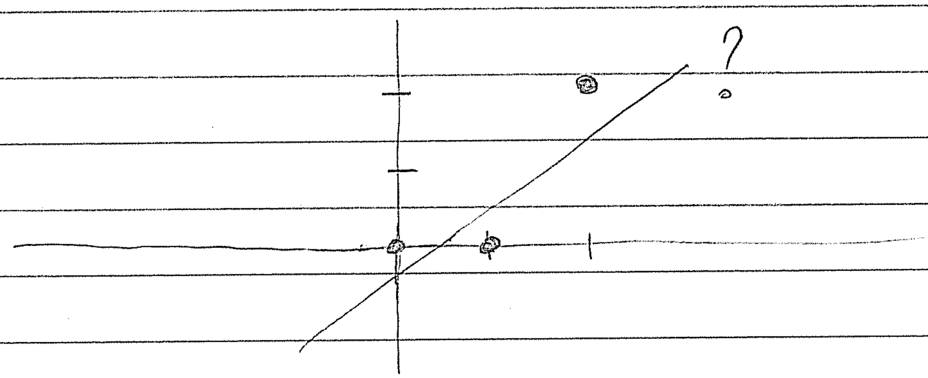
$$\boxed{\text{Matrix Multiplication} = \text{Composition of Functions}}$$

Mon Mar 4

Exam 1 is over.
Now we turn to Applications

Consider the three points:

$$\begin{pmatrix} t \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$



Do they lie on a line? Let's see.

Suppose the line $C + tD = b$ contains the three points, so

$$\begin{cases} C + 0D = 0 \\ C + 1D = 0 \\ C + 2D = 2 \end{cases}$$

Solve for C and D.

Try:

$$\begin{aligned} C + 0 &= 0 \\ C + D &= 0 \\ C + 2D &= 2 \end{aligned} \quad \longrightarrow \quad \begin{aligned} C + 0 &= 0 \\ 0 + D &= 0 \\ 0 + 2D &= 2 \end{aligned}$$

$$\longrightarrow \quad \begin{aligned} C + 0 &= 0 \\ 0 + D &= 0 \\ 0 + 0 &= 2 \end{aligned}$$

$\times$ NO SOLUTION.

We're not surprised.

In matrix form,

$$A\vec{x} = \vec{b}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} C \\ D \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

$$A\vec{x} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = C \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + D \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

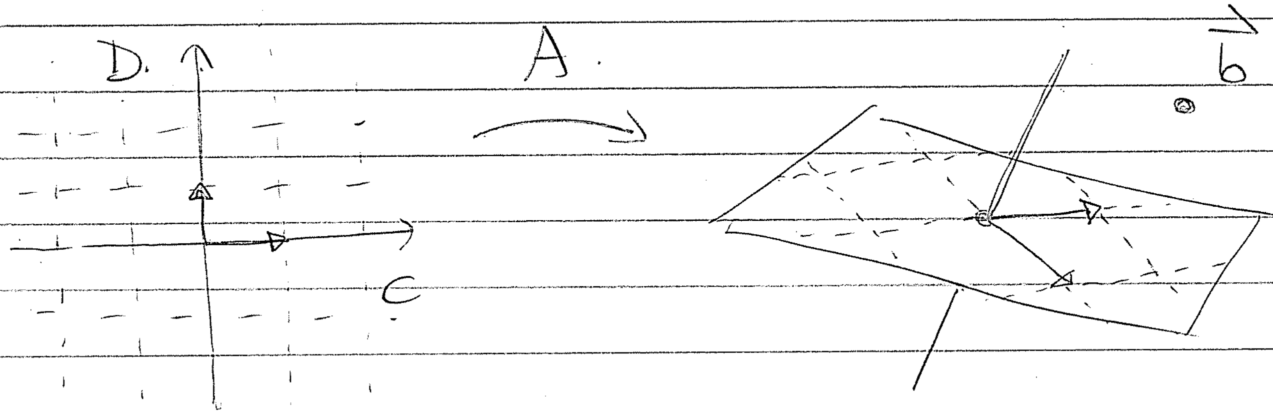
We can think $\mathbb{R}^2 \xrightarrow{A} \mathbb{R}^3$

$$\begin{pmatrix} C \\ D \end{pmatrix} \longrightarrow C\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + D\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

The matrix A sends all of $\mathbb{R}^2$ onto the <u>plane</u>

$$C\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + D\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \text{ in } \mathbb{R}^3.$$

This is called the <u>image</u> of the function, or the <u>column space</u> of matrix A.



So what's the problem? The target point $\vec{b} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$ is NOT in the image of A!

Image of A is all vectors of the form

$$A\vec{x} = C\begin{pmatrix}1\\1\\1\end{pmatrix} + D\begin{pmatrix}0\\1\\2\end{pmatrix},$$

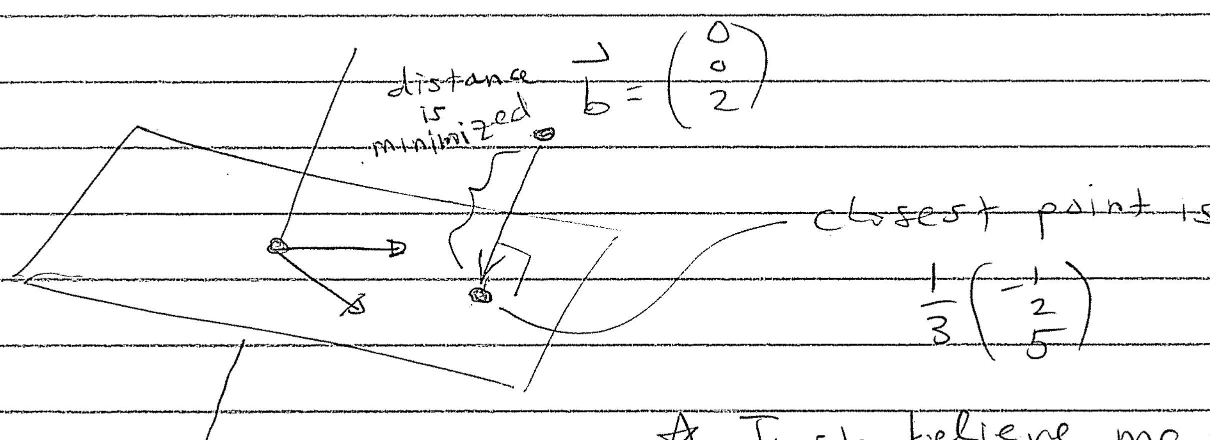
But $A\vec{x} = \vec{b}$ has no solution, i.e. $\vec{b}$ is not in the image of A.

Here is Gauss' Big Idea (1795; he was 18 years old).

"Least Squares Approximation":

Find the point in the image $A\vec{x}$ that is closest to $\vec{b}$

In our case we get



☆ Just believe me ☆
for now ...

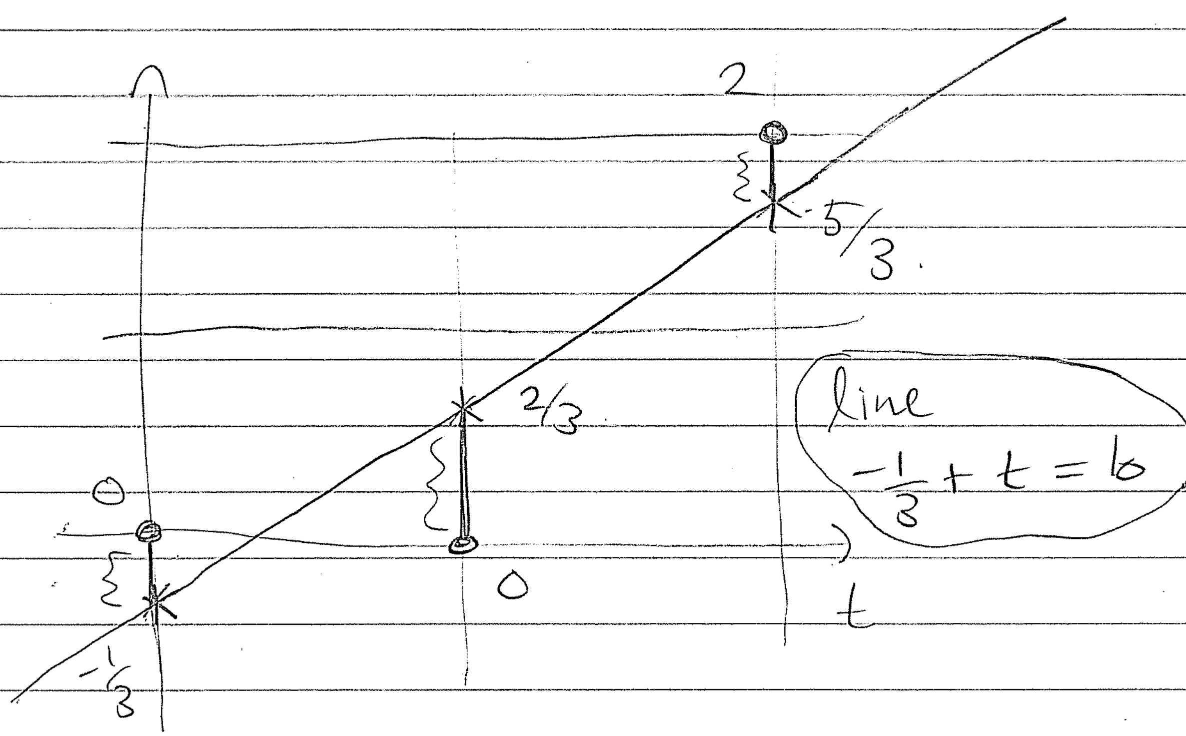
So this equation <u>does</u> have a solution:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} -1/3 \\ 2/3 \\ 5/3 \end{pmatrix}.$$

Find it

$$\left(\begin{array}{cc|c} 1 & 0 & -1/3 \\ 1 & 1 & 2/3 \\ 1 & 2 & 5/3 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & -1/3 \\ 0 & 1 & 1 \\ \cancel{0} & \cancel{2} & \cancel{2} \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & 0 & -1/3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow \begin{array}{l} C = -1/3 \\ D = 1 \end{array}$$



What was minimized? The distance.

$$\left\| \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} -1/3 \\ 2/3 \\ 5/3 \end{pmatrix} \right\|^2$$

$$= \left(0 - \frac{-1}{3}\right)^2 + \left(0 - \frac{2}{3}\right)^2 + \left(2 - \frac{5}{3}\right)^2$$

= sum of squares of vertical errors.

HOW did we do it?

That's our new topic:

ORTHOGONAL PROJECTION.

Wed Mar 6

HW 5 Average 26.5/30.
HW 6 will be due Fri Mar 22.
We will (mostly) skip Chapter 3.

Today: Orthogonal Projection.

First define the transpose A^T of a matrix A.

Def:

i,j entry of A^T = j,i entry of A.

Example

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

So what?
This gives us a nice way to write dot products.

Given $n \times 1$ column vectors $\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

we have.

$$\vec{u}^T \vec{v} = \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u_1 v_1 + \cdots + u_n v_n.$$

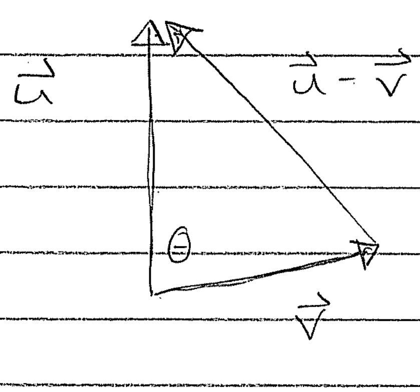
$1 \times n$ —— $n \times 1$ $\qquad$ 1×1

row $\qquad$ col. $\qquad$ number.

This is just the dot product

$$\boxed{\vec{u}^T \vec{v} = \vec{u} \cdot \vec{v}}$$

Given vectors $\vec{u}, \vec{v}$ with



Recall that

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\| \cdot \|\vec{v}\| \cos \theta.$$

But also ...

$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})^T(\vec{u}-\vec{v})$$

$$= (\vec{u}^T - \vec{v}^T)(\vec{u}-\vec{v})$$

$$= \vec{u}^T\vec{u} - \vec{u}^T\vec{v} - \vec{v}^T\vec{u} + \vec{v}^T\vec{v}$$

$$= \vec{u}^T\vec{u} + \vec{v}^T\vec{v} - 2\vec{u}^T\vec{v} \qquad [\vec{u}^T\vec{v} = \vec{v}^T\vec{u}]$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u}^T\vec{v}$$

We conclude that

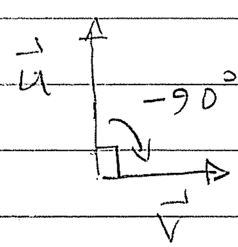
$$\vec{u}^T\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos\theta$$

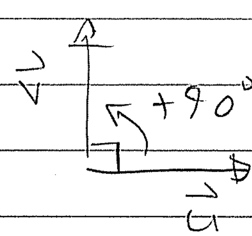
Hence

$\vec{u}$ and $\vec{v}$ are orthogonal

$$\Updownarrow$$

θ is $\pm 90^\circ$





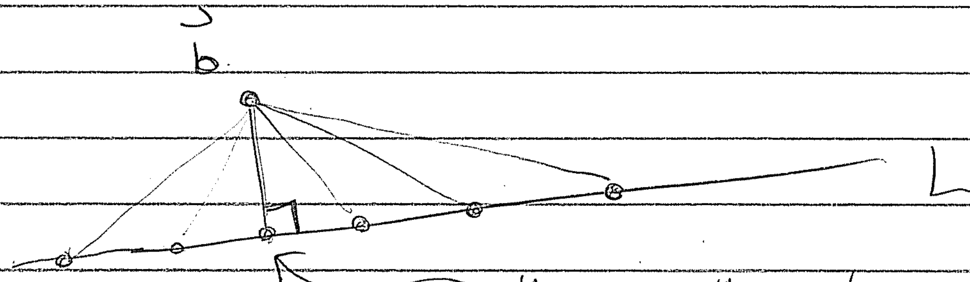
$$\Updownarrow$$

$$\cos\theta = 0$$

$$\Updownarrow$$

$$\vec{u}^T\vec{v} = 0$$

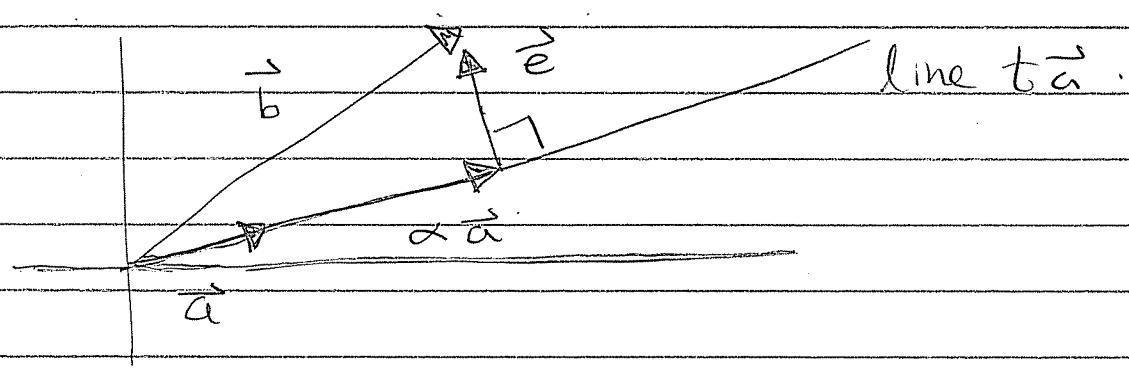
Geometry Problem: Given a line L and a point $\vec{b}$, find the point on L closest to $\vec{b}$



this is the closest.

The answer is the orthogonal projection of $\vec{b}$ onto L.

Compute the projection of $\vec{b}$ onto the line $t\vec{a}$.



The answer is $\alpha\vec{a}$ for some number α

Solve for α.

(Do we have an equation?)

Here's an equation:
We want $\vec{a}$ orthogonal to the "error vector" $\vec{e} = \vec{b} - \alpha\vec{a}$. Hence

$$\vec{a}^T(\vec{b} - \alpha\vec{a}) = 0.$$

$$\vec{a}^T\vec{b} - \alpha\,\vec{a}^T\vec{a} = 0.$$

$$\vec{a}^T\vec{b} = \alpha\,\vec{a}^T\vec{a}.$$

$$\alpha = \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}} = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}.$$

Summary: The projection of $\vec{b}$ onto the line spanned by $\vec{a}$ is

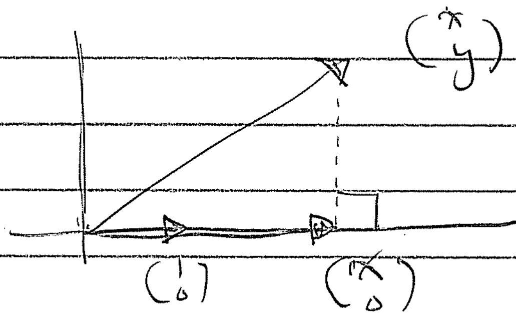
$$\mathrm{proj}_{\vec{a}}(\vec{b}) = \left(\frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}\right)\vec{a}$$

Examples:

1. Project $\begin{pmatrix} x \\ y \end{pmatrix}$ onto $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$\mathrm{proj}_{\begin{pmatrix}1\\0\end{pmatrix}}\begin{pmatrix}x\\y\end{pmatrix} = \left(\frac{(1\ 0)\begin{pmatrix}x\\y\end{pmatrix}}{(1\ 0)\begin{pmatrix}1\\0\end{pmatrix}}\right)\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \frac{x}{1}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}x\\0\end{pmatrix}$$



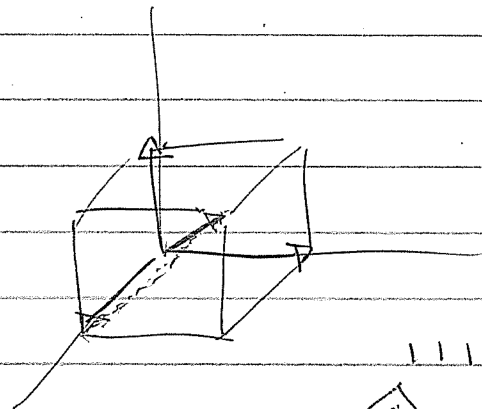
NO SURPRISE.

(2) Project $\begin{pmatrix}1\\0\\0\end{pmatrix}$ onto $\begin{pmatrix}1\\1\\1\end{pmatrix}$

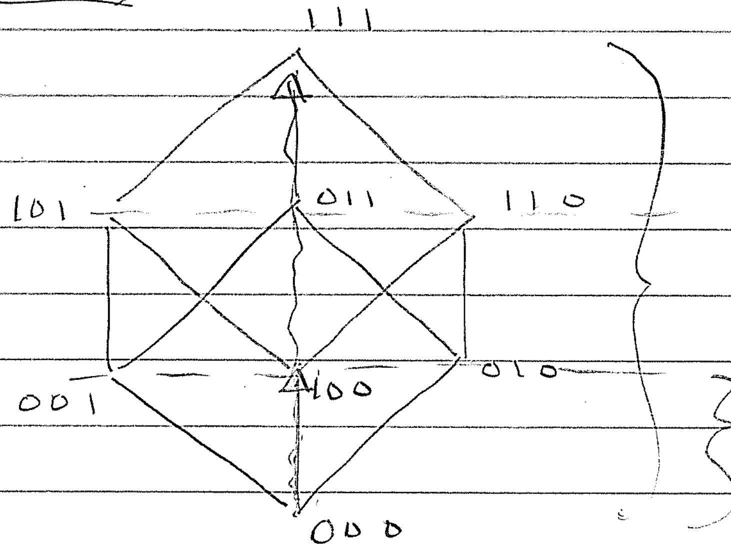
$$\mathrm{proj}_{\begin{pmatrix}1\\1\\1\end{pmatrix}}\begin{pmatrix}1\\0\\0\end{pmatrix} = \left(\frac{(1\ 1\ 1)\begin{pmatrix}1\\0\\0\end{pmatrix}}{(1\ 1\ 1)\begin{pmatrix}1\\1\\1\end{pmatrix}}\right)\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

Picture:



Better Picture



$\frac{1}{3}$ of the way up to $\begin{pmatrix}1\\1\\1\end{pmatrix}$.