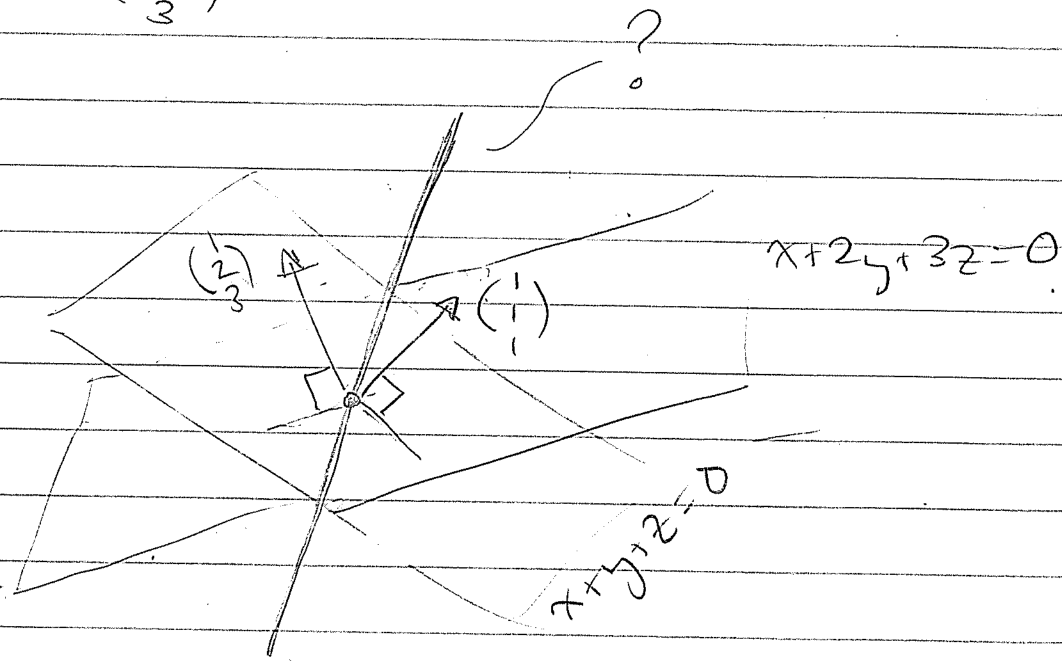
Mon Jan 28

~ 2 due Friday

Office Hours Today 2 — 3

Problem: Find a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ that is SIMULTANEOUSLY orthogonal to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Picture.



We expect a <u>line</u> of solutions.

We want

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \quad \& \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0$$

$$x+y+z=0 \quad \& \quad x+2y+3z=0.$$

We want to solve the system of two equations

$$\begin{cases} x + y + z = 0 & (1) \\ x + 2y + 3z = 0 & (2) \end{cases}$$

HOW? Try to simplify by adding and subtracting equations.

Cancel x with (2) − (1)

$$\begin{array}{r} (x + 2y + 3z = 0) \\ -\ (x + y + z = 0) \\ \hline y + 2z = 0 \end{array} \qquad (3)$$

Cancel y with (1) − (3)

$$\begin{array}{r} (x + y + z = 0) \\ -\ (0 + y + 2z = 0) \\ \hline x + 0 - z = 0 \end{array} \qquad (4)$$

Consider equations (3) and (4)

$$\begin{cases} x - z = 0 & (3) \\ y + 2z = 0 & (4) \end{cases}$$

That's pretty good :)

Let $z = t$ be a parameter. Then

(3) $\Longrightarrow$ $x = z = t$.

(4) $\Longrightarrow$ $y = -2z = -2t$. Hence

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ -2t \\ t \end{pmatrix} = t \underbrace{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}.$$

The line in 3D spanned by $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

Any t gives a solution.

E.g. let $t = 1$, so $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Check:

$$\begin{pmatrix}1\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} = 1\cdot 1 + (-2)\cdot 1 + 1\cdot 1 = 1-2+1 = 0 \checkmark$$

$$\begin{pmatrix}1\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\end{pmatrix} = 1\cdot 1 + (-2)\cdot 2 + 1\cdot 3 = 1-4+3 = 0 \checkmark$$

[If you are familiar with the "cross product" in 3D

$$\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}\times\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{pmatrix},$$

observe that

$$\begin{pmatrix}1\\1\\1\end{pmatrix}\times\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}3-2\\1-3\\2-1\end{pmatrix} = \begin{pmatrix}1\\-2\\1\end{pmatrix}$$

Is it a coincidence? No.]

Another point of view:

We can rewrite the system of two equations

$$\begin{cases} x + y + z = 0 \\ x + 2y + 3z = 0 \end{cases}$$

as a single vector equation:

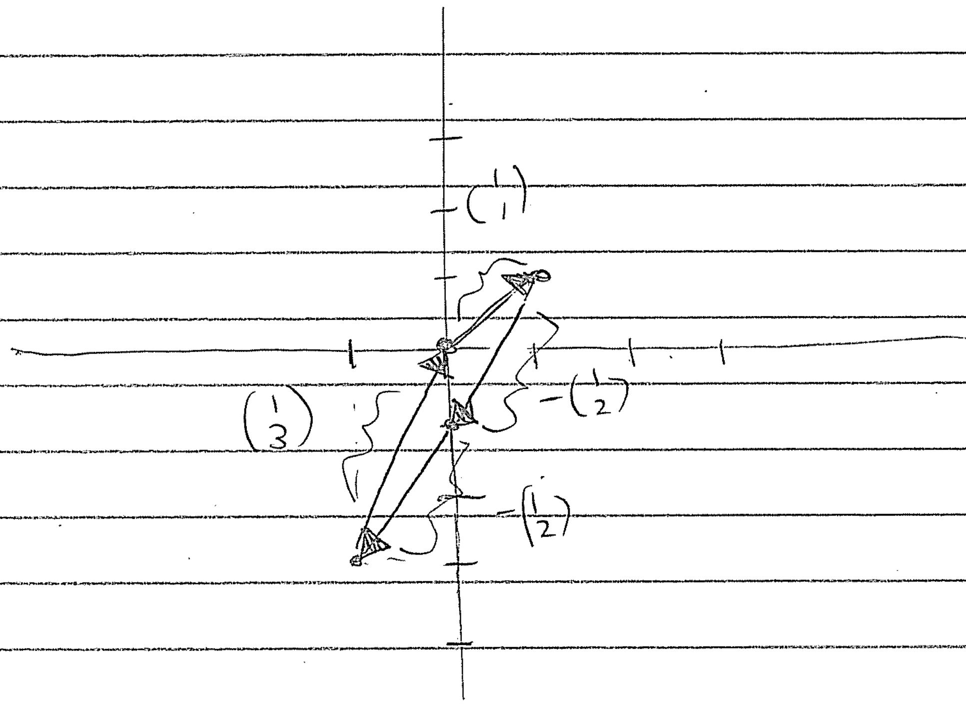
$$\begin{pmatrix} x + y + z \\ x + 2y + 3z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underbrace{x\begin{pmatrix} 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ 2 \end{pmatrix} + z\begin{pmatrix} 1 \\ 3 \end{pmatrix}}_{\substack{\text{a "linear combination"} \\ \text{of the vectors } \binom{1}{1}, \binom{1}{2}, \binom{1}{3} \text{ in 2D.}}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Problem: Express $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Solution: We already know that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \text{ works.}$$



$$1\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is the "column picture" of the system.

Q: Given n "constant" numbers

$a_1, a_2, \cdots, a_n$

and n "unknown" numbers

$x_1, x_2, \cdots, x_n$,

what does the equation

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

represent geometrically?

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = 0.$$

The set of vectors in nD space orthogonal to $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

form a $\underbrace{\text{"hyperplane"}}_{\text{it is } (n-1)\text{-dimensional}}$.

Wed Jan 30

Exams Fri Mar 1
Fri Apr 19
Final Mon May 6

HW 2 due Friday
Office Hours Today 3-4

Recall: Given a vector $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ in n-dimensional space, the set of vectors $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ such that

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b.$$

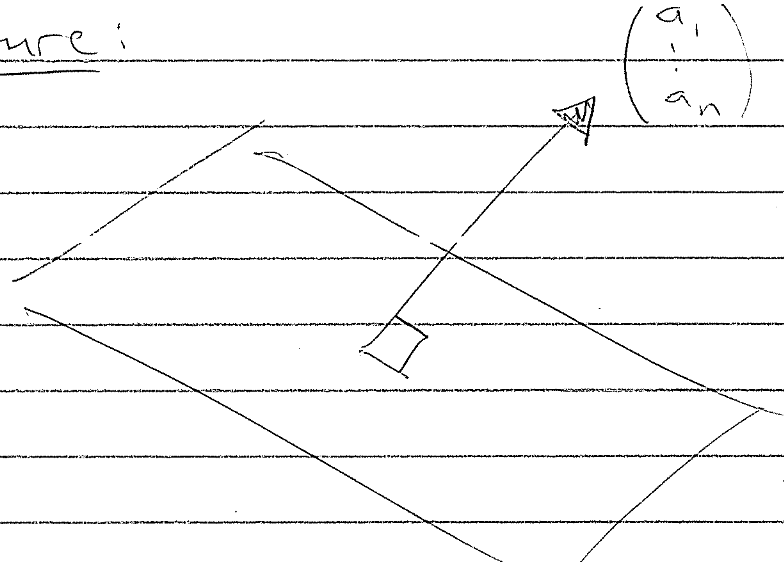
some number

this is called a "linear" equation

$$\boxed{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b}$$

form an (n-1)-dimensional "hyperplane" orthogonal to $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

Picture:



The central problem of Linear Algebra is to solve a system of m linear equations in n unknows

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

There are two points of view.

(1) The "Row Picture"

Compute the intersection of m hyperplanes in n-dimensional space.

Intuition: If the equations are "random" or "generic", the solution will have

n - m dimensions.

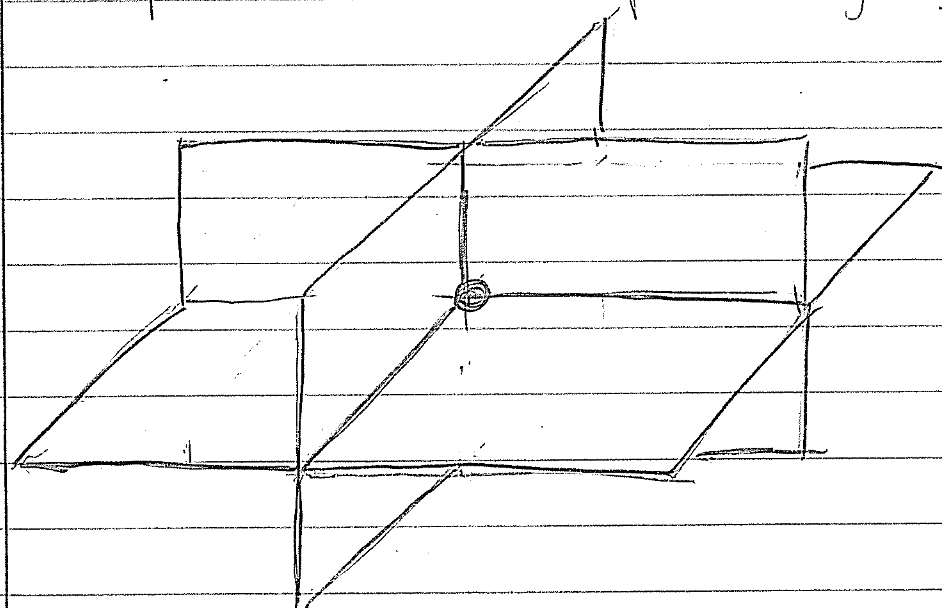
(If $m > n$, there is "probably" NO SOLUTION)

$m=3$ $n=3$.

eg 3 planes in 3D probably meet in a point.

A point has $0 = n-m = 3-3$ dimensions

4 planes in 3D probably don't meet.



(2) The "Column Picture"

Use vector notation

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n = \vec{b}.$$

Problem: Combine n given vectors in m-dimensional space to reach a given <u>target</u> vector.

Completely equivalent to the row picture, just a different interpretation.

Q: If I give you 3 vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ in 4D space and a target vector $\vec{b}$,

can you find numbers x_1, x_2, x_3 such that

$$x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b} \quad ?$$

A: PROBABLY NOT.

Because the vectors $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3$ form a 3D hyperplane, and vector $\vec{b}$ is probably NOT in that hyperplane.

The Matrix Notation:

We can encode the system very simply as

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{\text{called a "matrix"}} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\boxed{A\vec{x} = \vec{b}}$$

matrix * vector = vector.

Example:

$$\begin{cases} x + 2y + 3z = 6 \\ 2x + 5y + 2z = 4 \\ 6x - 3y + z = 2 \end{cases}$$

$$x\begin{pmatrix}1\\2\\6\end{pmatrix} + y\begin{pmatrix}2\\5\\-3\end{pmatrix} + z\begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}6\\4\\2\end{pmatrix}$$

$$\underbrace{\begin{pmatrix}1 & 2 & 3\\2 & 5 & 2\\6 & -3 & 1\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}x\\y\\z\end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix}6\\4\\2\end{pmatrix}}_{\vec{b}}$$

$$A\vec{x} = \vec{b}$$

Matrix notation suggests an <u>idea</u>:

Given numbers a, b and a variable, solve for x:

$$ax = b.$$

If $a \neq 0$ we can divide both sides by a:

$$x = \frac{b}{a}.$$

Analogy: Given matrix A and vector $\vec{b}$, solve for vector $\vec{x}$:

$$A\vec{x} = \vec{b}$$

If "$A \neq 0$", maybe we can "divide" both sides by A ...

$$A\vec{x} = \vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

Yes, we will do this, but it's more complicated than you think.

Fri Feb 1

HW 2 due NOW
HW 3 due next Friday.

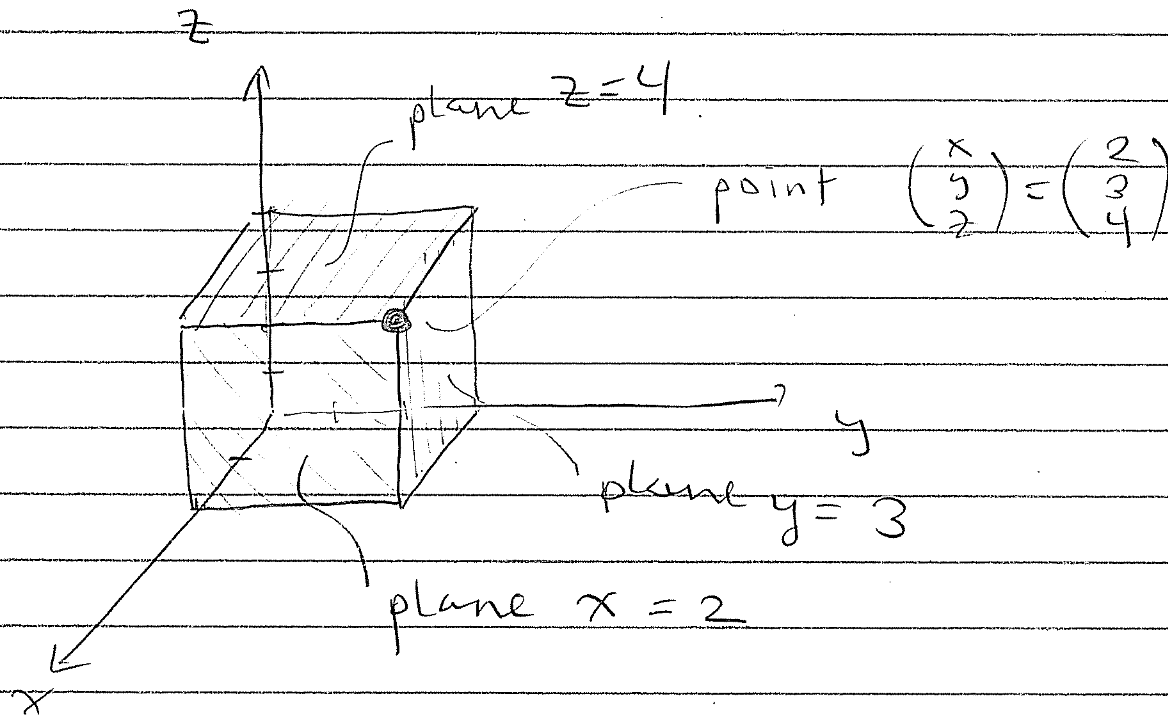
In Class Exams:

Fri Mar 1
Fri Apr 19

Final: Mon May 6

Today: Discuss HW 2

How to draw a box:
Edges are parallel to axes



The planes $x=2$, $y=3$, $z=4$ meet in the point $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

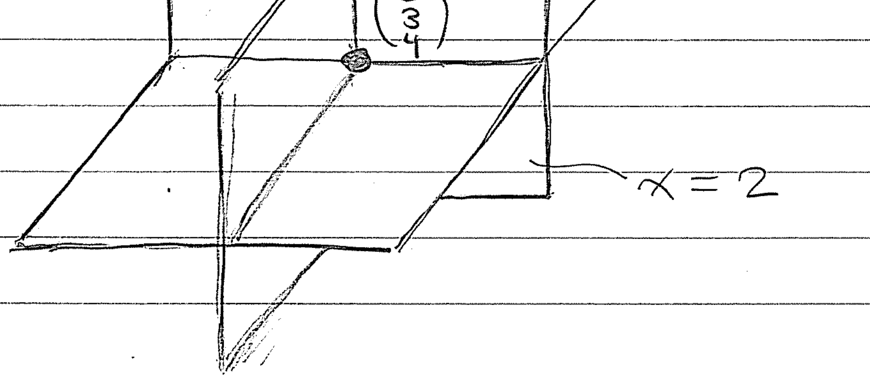
Consider 3×3 system:

$$\begin{cases} x + 0y + 0z = 2 \\ 0x + y + 0z = 3 \\ 0x + 0y + z = 4 \end{cases}$$

$$x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

Row Picture:

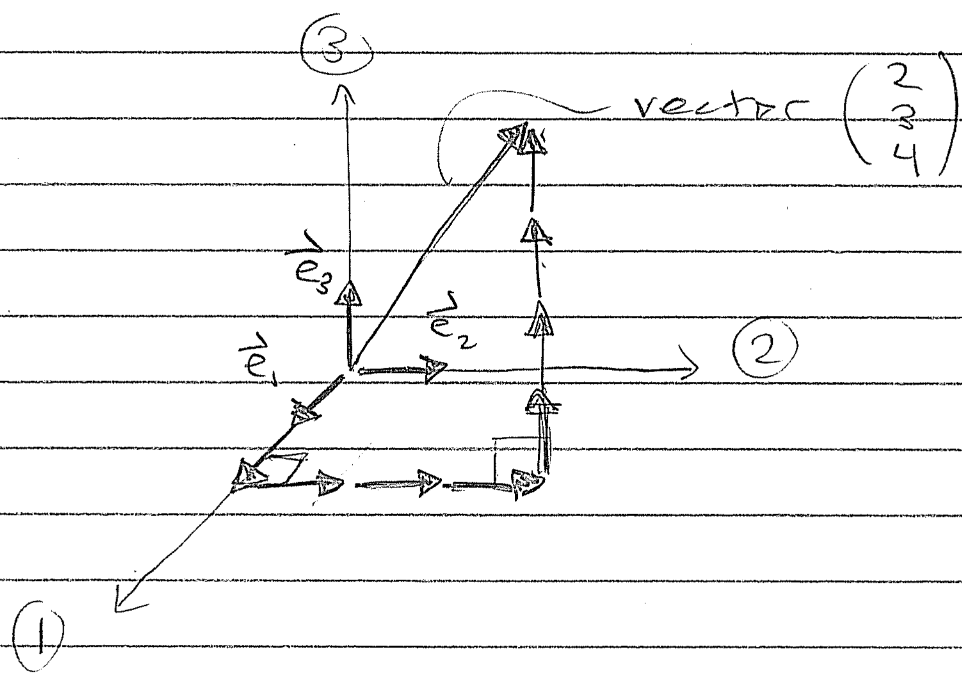


Column Picture:

$$2\begin{pmatrix}1\\0\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\\0\end{pmatrix} + 4\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}2\\3\\4\end{pmatrix}$$

$$2\vec{e}_1 + 3\vec{e}_2 + 4\vec{e}_3 = \begin{pmatrix}2\\3\\4\end{pmatrix}$$

Show the vector addition



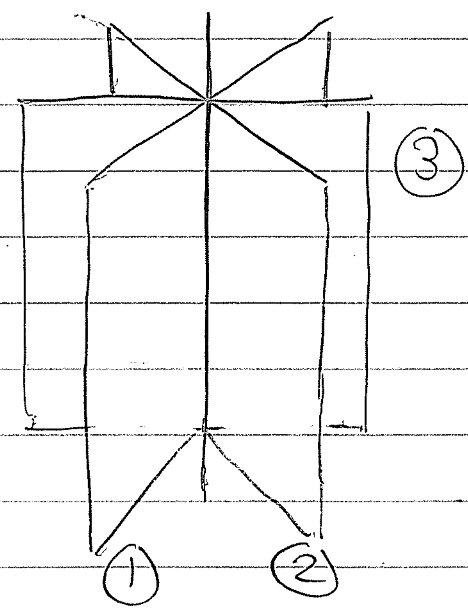
Next consider

$$\begin{cases} x + y + z = 2 & ① \\ x + 2y + z = 3 & ② \\ 2x + 3y + 2z = 5 & ③ \end{cases}$$

Note ① + ② = ③

$$\begin{array}{rl} & x + y + z = 2 \\ + & x + 2y + z = 3 \\ \hline & 2x + 3y + 2z = 5 \end{array}$$

Means: Intersection of planes ① & ② is contained in plane ③



Adding / Subtracting equations gives

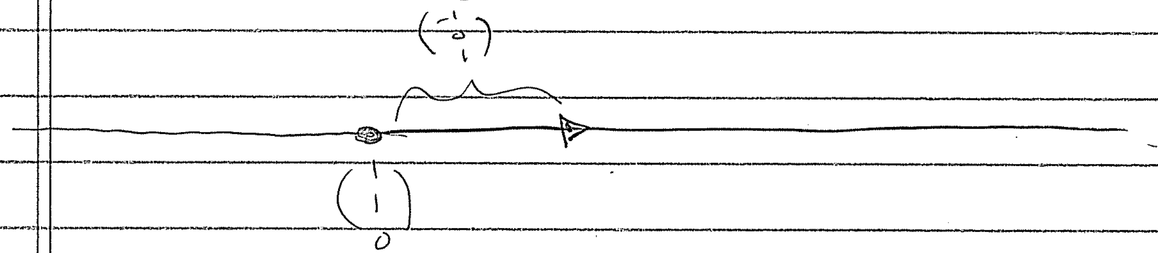
$$\begin{cases} x + 0y + z = 1 \\ 0x + y + 0z = 0 \\ 0x + 0y + 0z = 0 \end{cases}$$

Take $z = t$ as a parameter. Then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-t \\ 1 \\ t \end{pmatrix} = \begin{pmatrix} 1 - 1t \\ 1 + 0t \\ 0 + 1t \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The line through $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ spanned by $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$



Any choice of t gives a solution

$t = -1$, $t = 0$, $t = 1$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

There are ∞ many more . . .

Solve the system

$$\begin{cases} x + y + z = 4 & (1) \\ x + 2y + z = 6 & (2) \\ 2x + 3y + 2z = 10 & (3) \end{cases}$$

Scaled by 2

Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x' \\ 2y' \\ 2z' \end{pmatrix}$. Then substitute:

$$\begin{cases} x' + y' + z' = 2 \\ x' + 2y' + z' = 3 \\ 2x' + 3y' + 2z' = 4 \end{cases}$$

$$\Longrightarrow \quad \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Longrightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$$

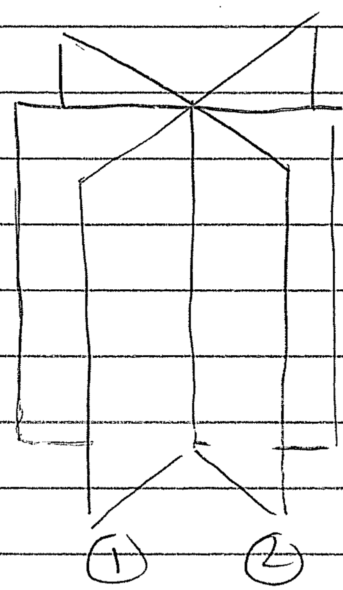
The solution scaled by 2

Finally:

$$\begin{cases} x + y + z = 4 & \text{(1)} \\ x + 2y + z = 6 & \text{(2)} \\ 2x + 3y + 2z = c = \text{unknown} & \text{(3)*} \end{cases}$$

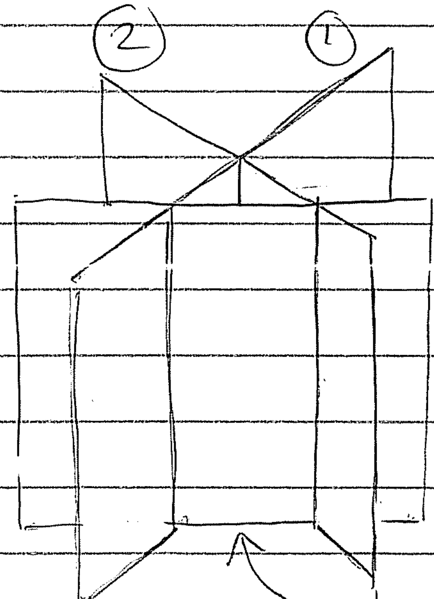
Note (3) & (3)* are <u>parallel</u>.

Picture:



(3) $(c = 10)$.

solution is a line



(3)* $(c \neq 10)$

NO SOLUTION!

triangular hole in the middle.