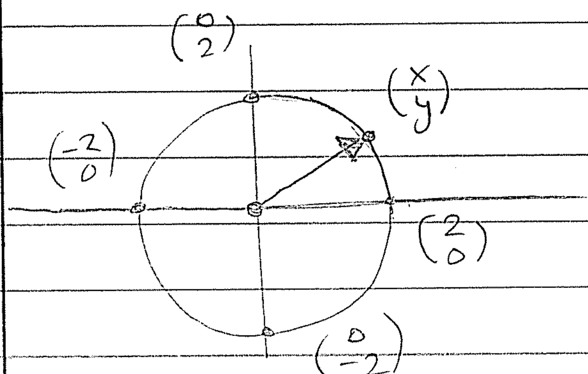
Wed Jan 23

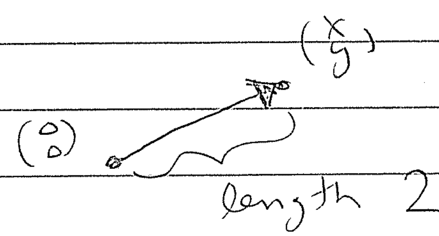
HW 1 due at the
BEGINNING of Friday's class.

Office Hours today 3 – 4 pm
in Ungar 533.

Recall:
Cartesian coordinates allow us to
turn Geometry into Algebra.

e.g. Consider the circle of radius 2
centered at the origin:





If $\binom{x}{y}$ is any point on the circle
then we have

$$\left\| \binom{x}{y} \right\| = 2$$

$$\Longrightarrow \quad \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 = 4.$$

$$\Longrightarrow \quad \boxed{x^2 + y^2 = 4}$$ "The equation of the circle."

The circle *is* the set of points $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $x^2 + y^2 = 4$.

Q: What is $2x + y = 0$ geometrically?

Rewrite $2x + y = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$.

Then $\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \left\| \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\| \cdot \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| \cos \theta$

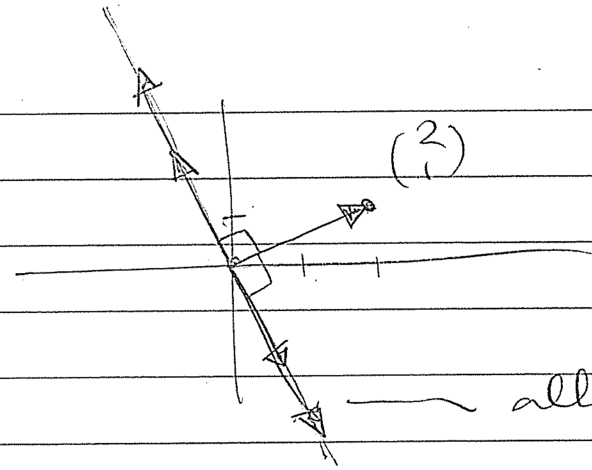
$= 0$

means that $\theta = 90°$.

We say $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix}$ are perpendicular (or orthogonal)

and write $\begin{pmatrix} 2 \\ 1 \end{pmatrix} \perp \begin{pmatrix} x \\ y \end{pmatrix}$

Picture:



So $2x + y = 0$ is the line containing $\binom{0}{0}$ and perpendicular to $\binom{2}{1}$

More Explicit: Let $y = t$ be a parameter.
Then

$$2x + y = 0$$
$$2x = -y$$
$$x = -y/2 = -t/2.$$

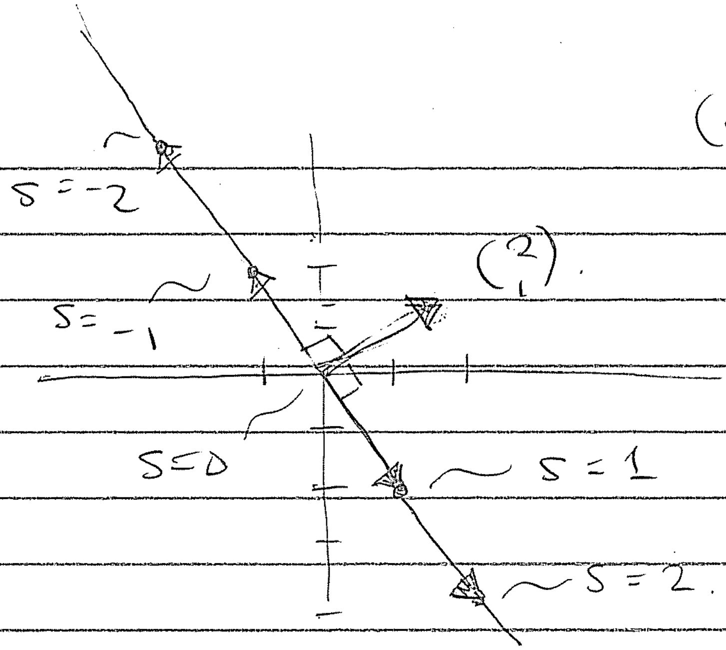
So $$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -t/2 \\ t \end{pmatrix} = t \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}.$$

If we want, let $s = -t/2$, so $t = -2s$.
Then

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -t/2 \\ t \end{pmatrix} = \begin{pmatrix} s \\ -2s \end{pmatrix} = s \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

the parameter s ranges over all numbers.

(sorry not to scale)



$\begin{pmatrix} x \\ y \end{pmatrix} = s \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is the line <u>span</u>ned by $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, <u>parametrized</u> by s

$$\boxed{\begin{aligned} x &= s \\ y &= -2s \end{aligned}} \longleftrightarrow \boxed{2x + y = 0}$$

same thing

Explicit

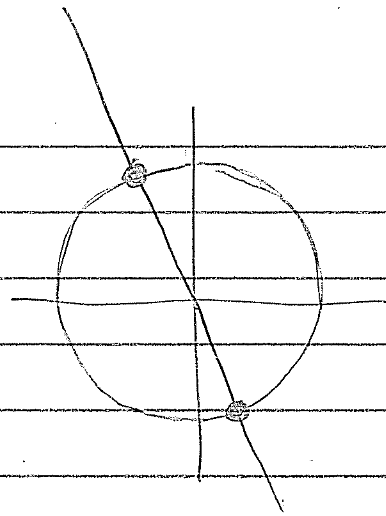
Implicit

Now, find all $\begin{pmatrix} x \\ y \end{pmatrix}$ such that

$$x^2 + y^2 = 4$$

<u>AND</u> $2x + y = 0$ simultaneously!

Just Guess!



There will be exactly two solutions

Simultaneous Equations = Intersection of Geometric Objects

Now compute :

① $x^2 + y^2 = 4$
② $2x + y = 0$.

② $\Rightarrow$ $y = -2x \Rightarrow y^2 = 4x^2$

Substitute into ① to get

$$x^2 + (4x^2) = 4$$
$$5x^2 = 4 \Rightarrow x^2 = 4/5.$$
$$x = \pm 2/\sqrt{5}.$$

Back-substitute into ②.

$$x = +2/\sqrt{5} \Rightarrow y = -4/\sqrt{5}$$
$$x = -2/\sqrt{5} \Rightarrow y = +4/\sqrt{5}$$

Two solutions :

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2/\sqrt{5} \\ -4/\sqrt{5} \end{pmatrix} = \frac{2}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

&

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2/\sqrt{5} \\ 4/\sqrt{5} \end{pmatrix} = -\frac{2}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

///

Q : What is $2x + y = 1$ geometrically ?

Use Brute Force : Let $y = t$ be a "free parameter".

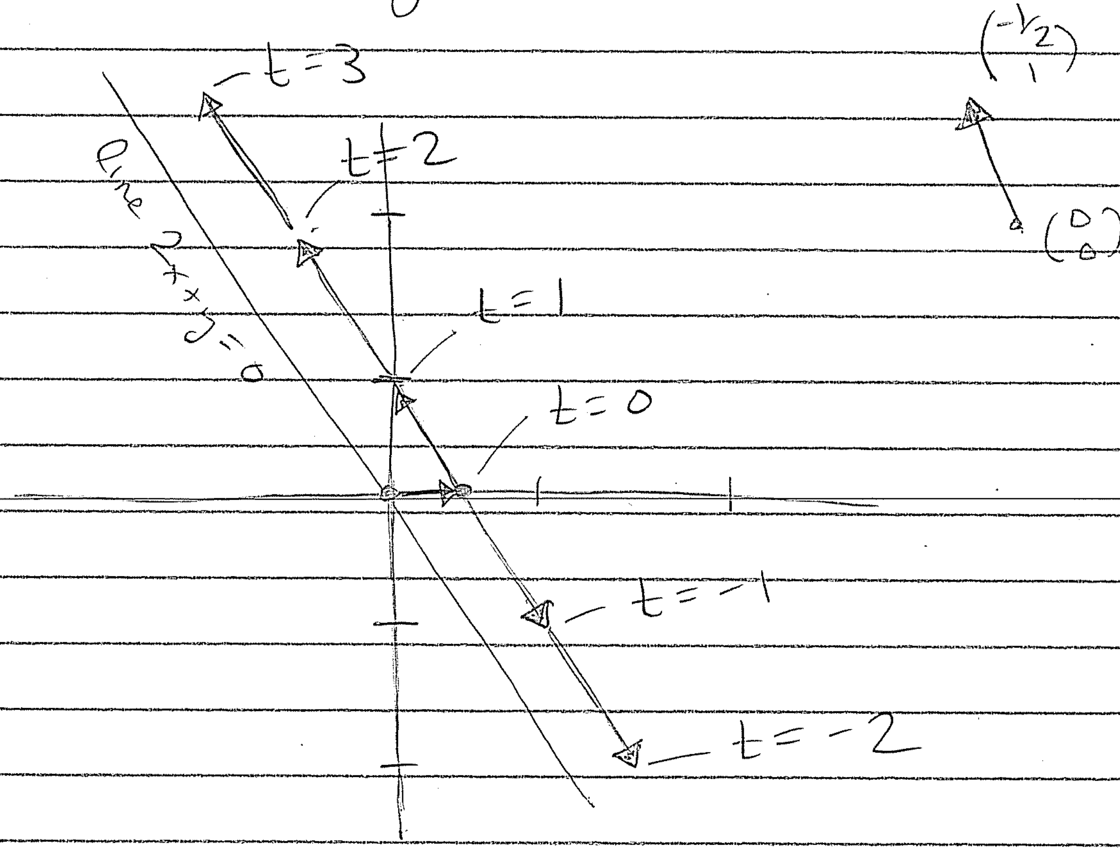
Then
$$2x + y = 1$$
$$2x = 1 - y$$
$$x = \frac{1}{2} - \frac{y}{2} = \frac{1}{2} - \frac{t}{2}$$

Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - \frac{t}{2} \\ t \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - \frac{t}{2} \\ 0 + t \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \begin{pmatrix} -t/2 \\ t \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$$

Picture: $\begin{pmatrix}x\\y\end{pmatrix} = \overbrace{\begin{pmatrix}1/2\\0\end{pmatrix} + t\begin{pmatrix}-1/2\\1\end{pmatrix}}^{\text{line } 2x+y=1}$



The line containing the point $\begin{pmatrix}1/2\\0\end{pmatrix}$

and moving in the direction $\begin{pmatrix}-1/2\\1\end{pmatrix}$

This line is parallel to $2x+y=0$

Fri Jan 25

HW 1 is due NOW.

HW 2 is due next Friday.

HW 2:
1.1 : 26
2.1 : 1, 4, 5, 6, 7, 8, 9, 13

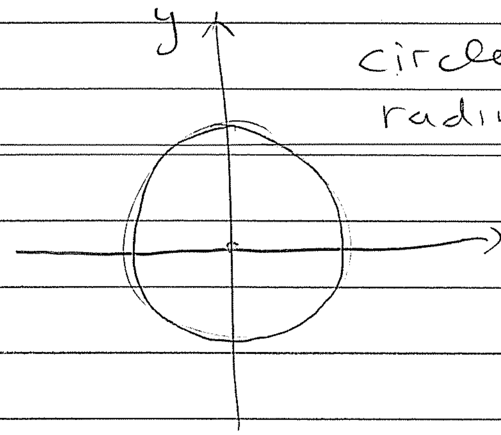
Recall:

Equation in n unknowns $\longleftrightarrow$ Geometric object in n-dimensional space.

Eg.

$$x^2 + y^2 = 4$$

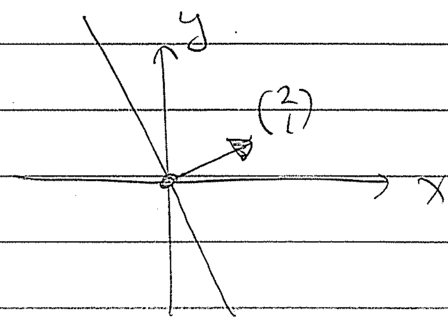
$$\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|^2 = 2^2$$



circle of radius 2

$$2x + y = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0.$$

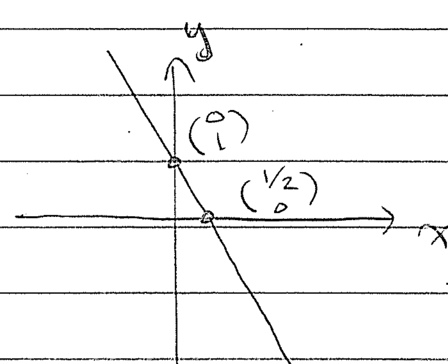


line through the origin perpendicular to $\binom{2}{1}$.

parallel

$$2x + y = 1$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 1.$$

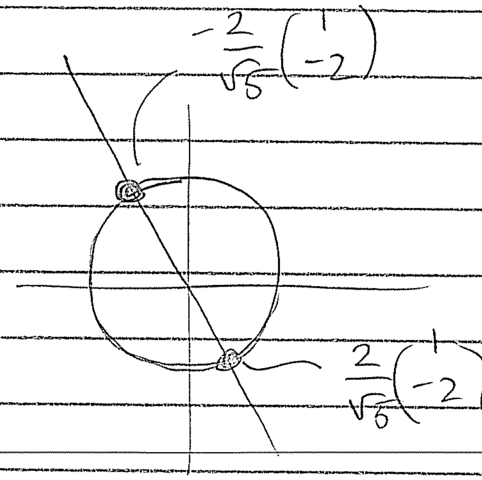


line containing points $\binom{0}{1}$ & $\binom{1/2}{0}$.

Simultaneous $\longleftrightarrow$ Intersection of
Equations Geometric objects

Eg.

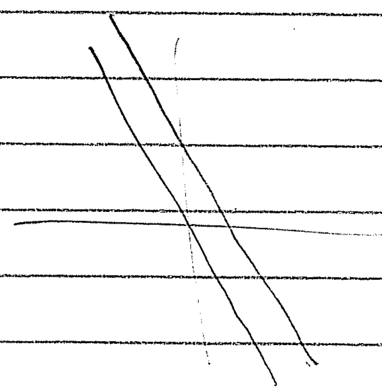
$$\begin{cases} x^2 + y^2 = 4 \\ 2x + y = 0 \end{cases}$$



Exactly two solutions:

$$\begin{pmatrix}x\\y\end{pmatrix} = \frac{2}{\sqrt{5}}\begin{pmatrix}1\\-2\end{pmatrix} \quad \text{OR} \quad -\frac{2}{\sqrt{5}}\begin{pmatrix}1\\-2\end{pmatrix}$$

$$\begin{cases} 2x + y = 0 \\ 2x + y = 1 \end{cases}$$



NO SOLUTION because
parallel lines don't intersect.

Another way to see this:

If $A = B$ and $C = D$, then

$$A + C = B + D \quad \text{and}$$
$$A - C = B - D$$

We can add/subtract true equations.

If $2x + y = 1$
and $2x + y = 0$ are both TRUE.

Then $(2x+y) - (2x+y) = 1 - 0$
$0 = 1$ is TRUE.

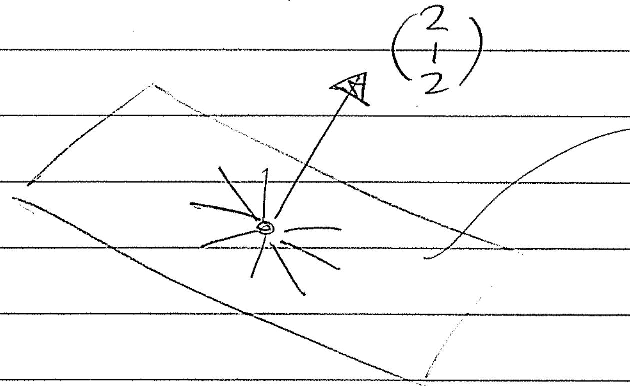
But clearly $0 = 1$ is FALSE.

Now move to 3D.

Q: What is $2x + y + 2z = 0$ geometrically?

A: A plane!

Picture : $2x + y + 2z = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 0.$



the plane of vectors perpendicular to $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

We need 2 parameters to describe it

Let's say $y = s$ & $z = t$.
(choice is completely arbitrary).

Then $2x + y + 2z = 0$

$$2x = -y - 2z.$$

$$x = -\frac{1}{2} y - z.$$

$$x = -\frac{1}{2} s - t.$$

Hence.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}s - t \\ s \\ t \end{pmatrix}$$

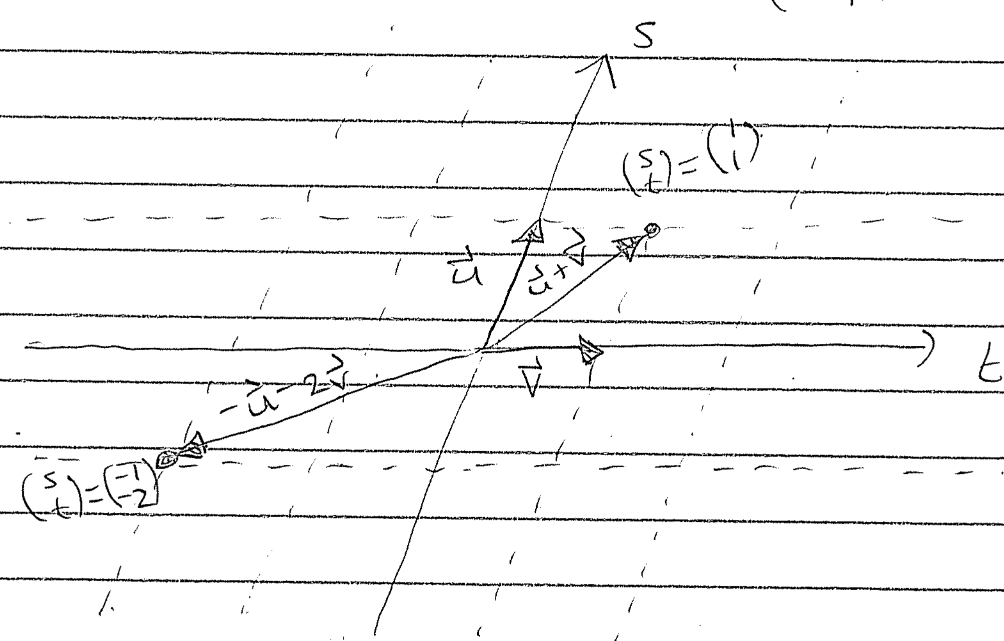
TRICK

$$= \begin{pmatrix} 0 - \frac{1}{2}s - t \\ 0 + s + 0t \\ 0 + 0s + t \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}s \\ s \\ 0s \end{pmatrix} + \begin{pmatrix} -t \\ 0t \\ t \end{pmatrix}$$

$$= \underbrace{s \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

the plane spanned (or generated) by $\vec{u} := \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{v} := \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$



We've defined a "coordinate system" or a "system of parameters" for the plane $2x + y + 2z = 0$.

Problem: Find the intersection of the planes $x+y+z=0$ and $x+2y+3z=0$

$$\begin{cases} x+y+z=0 & (1) \\ x+2y+3z=0 & (2) \end{cases}$$

Compute (2) − (1).

$$(\cancel{x}+2y+3z)-(\cancel{x}+y+z) = 0-0.$$

$$y+2z=0. \qquad (3)$$

Compute (1) − (3).

$$(x+\cancel{y}+z)-(\cancel{y}+2z) = 0-0.$$

$$x-z=0 \qquad (4)$$

We've simplified things :)

Look at equations (3) and (4)

$$\begin{cases} x \qquad - z = 0 \\ \quad y + 2z = 0 \end{cases}$$

Let $z = t$ be a parameter.

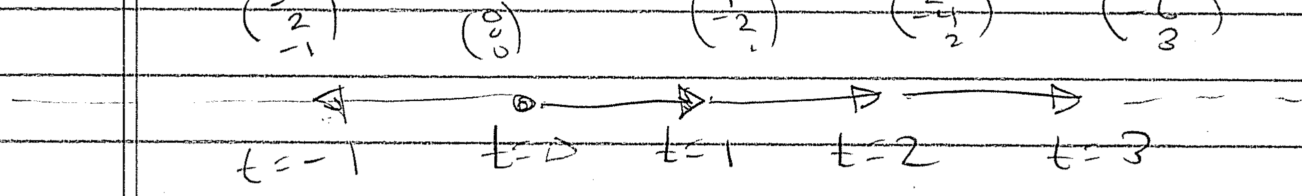
(3) $x - z = 0 \implies x = z = t$

(4) $y + 2z = 0 \implies y = -2z = -2t.$

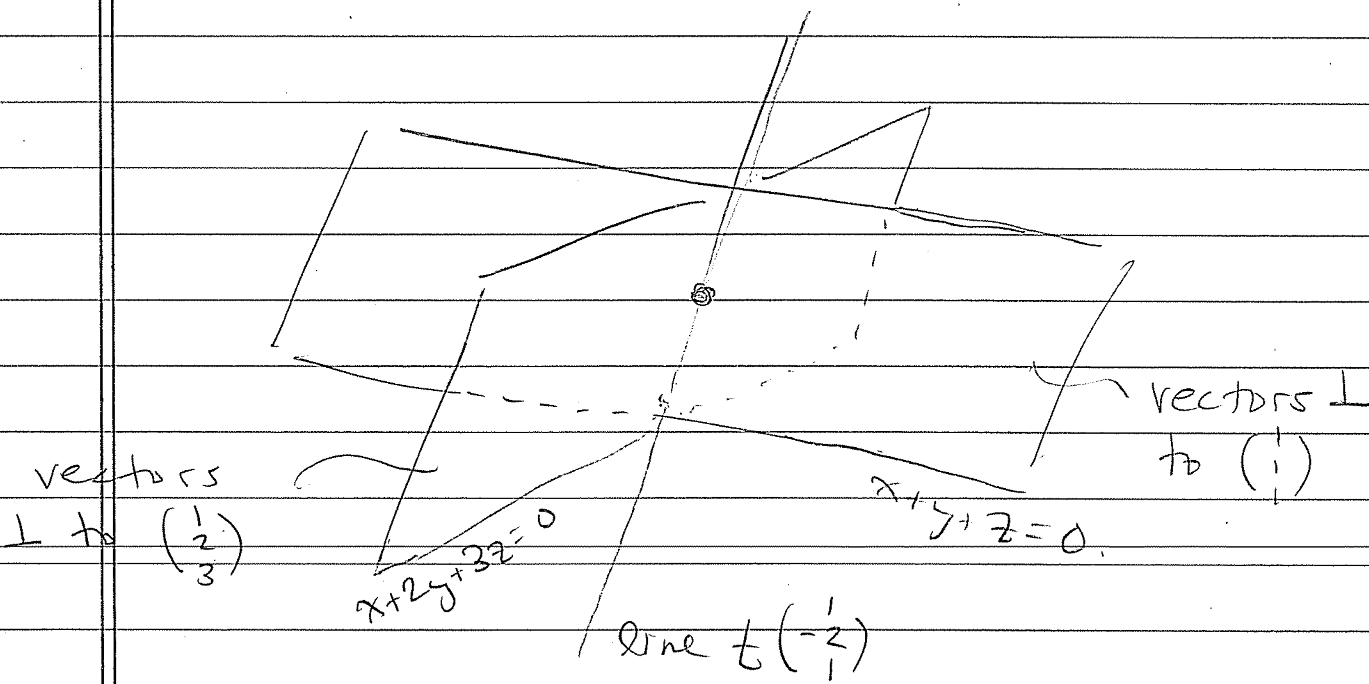
Hence

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ -2t \\ t \end{pmatrix} = t \underbrace{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}_{\text{What is this ?}}$$

The line in 3D space spanned (or generated) by $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.



What have we done? Picture.



Intersection of two planes is (usually) a line.

The vectors $t\begin{pmatrix}1\\-2\\1\end{pmatrix}$ are simultaneously

perpendicular to $\begin{pmatrix}1\\1\\1\end{pmatrix}$ AND to $\begin{pmatrix}1\\2\\3\end{pmatrix}$