Mon Jan 14
2013

MTH 210 F
Intro. to Linear Algebra.

Drew Armstrong
armstrong@math.miami.edu.

All course info will be on my webpage.

Text: Strang.

Evaluation

Homework	10%
Midterm 1	30%
Midterm 2	30%
Final Exam	30%.

Course:

1) Some geometry.
2) Static linear algebra
3) Dynamic linear algebra.

BEGIN

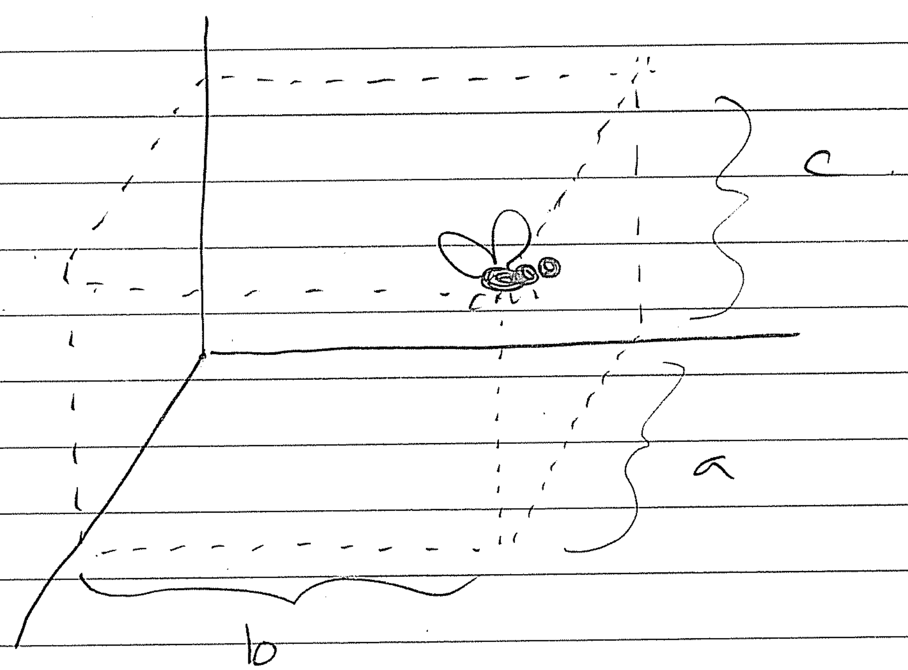
Q: What is "space"?
What is a "point"?

Answer (Fermat, Descartes ~1637):

A point is an ordered list of numbers

What?!

Descartes was lying in bed.
He saw a fly in the corner.



Imagine a rectangular box with dimensions a, b, c (in some order).

Descartes realized that the numbers (a, b, c) uniquely specify the position of the fly!

(a, b, c) = the "(Des) Cartesian coordinates" of the fly.

Actually, we'll write it like this:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \vec{v}$$

We'll call it a vector.
A vector is just an ordered list of numbers.

Q: Why is this useful?

point = list of numbers.

A: Because we can do algebra with numbers!

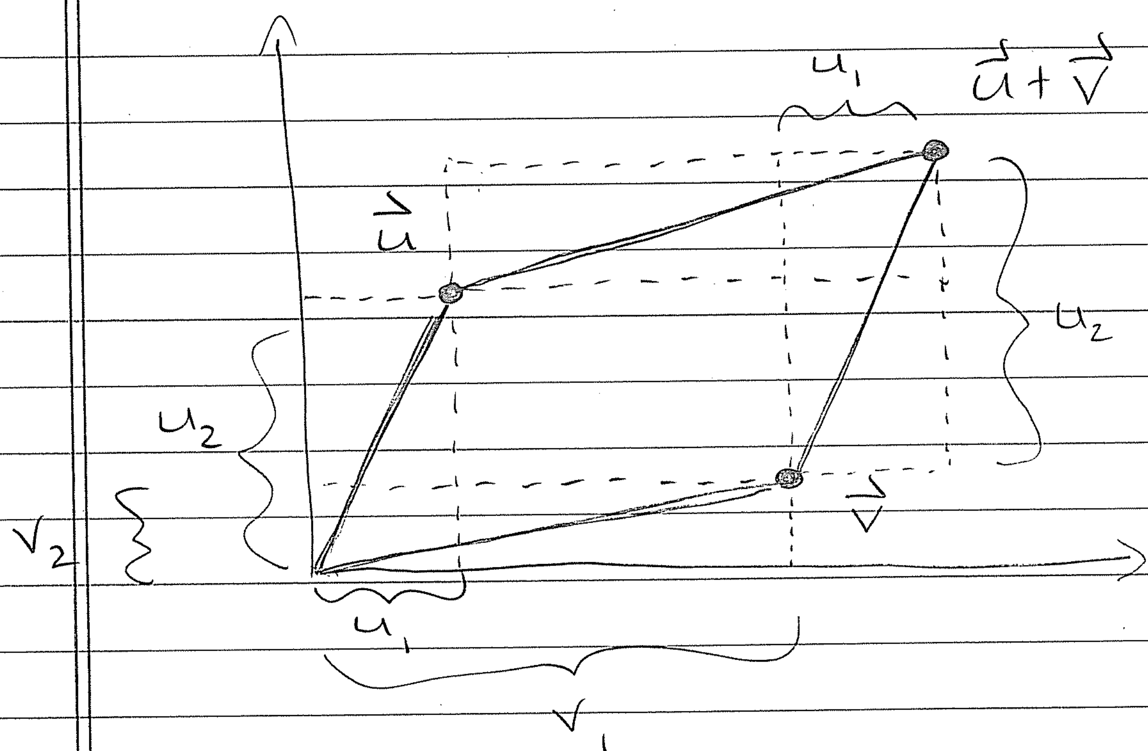
Example: We can add vectors

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

$$\vec{u} + \vec{v} := \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$$

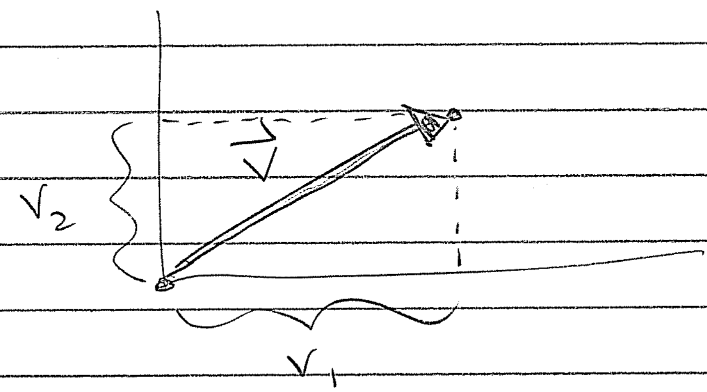
definition   add "componentwise"

What does it mean?

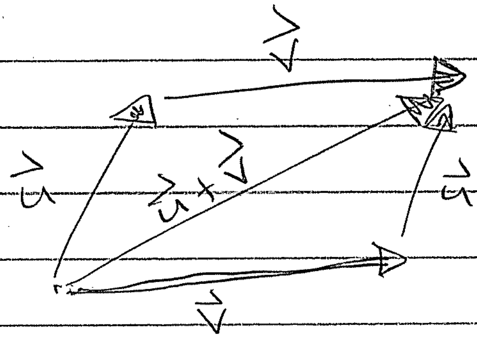


"Parallelogram Law of Vector Addition"

Sometimes useful to think of $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ as an arrow with head at $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and tail at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

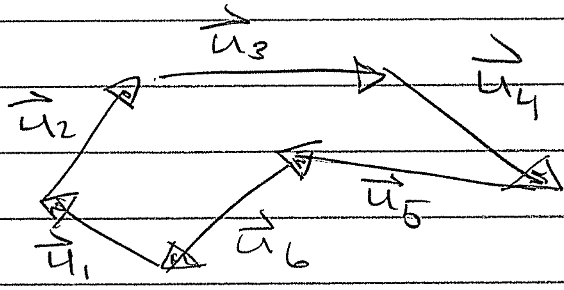


Then adding vectors is easy:



Order doesn't matter
$\vec{u}+\vec{v} = \vec{v}+\vec{u}$.

Vectors add head-to-tail.



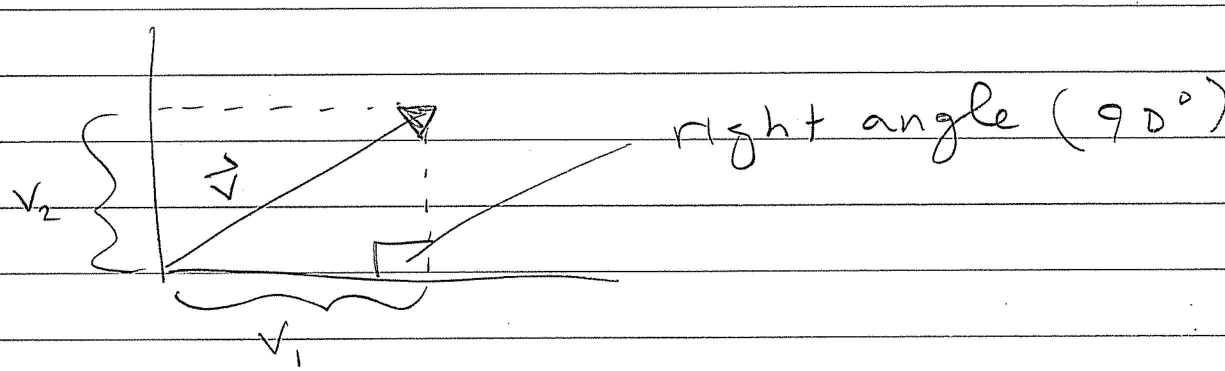
$\vec{u}_1+\vec{u}_2+\vec{u}_3$
$+\vec{u}_4+\vec{u}_5+\vec{u}_6$
$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0}$
"zero vector"

Two Perspectives:

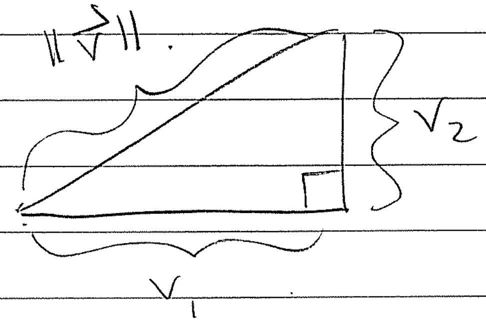
vector is a point $\longleftrightarrow$ vector is an arrow.

Switch back and forth when we want, using the point of reference $\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (the "origin").

Q: What is the <u>length</u> of a vector?



See the triangle?



Let $\|\vec{v}\| :=$ length of arrow $\vec{v}$.

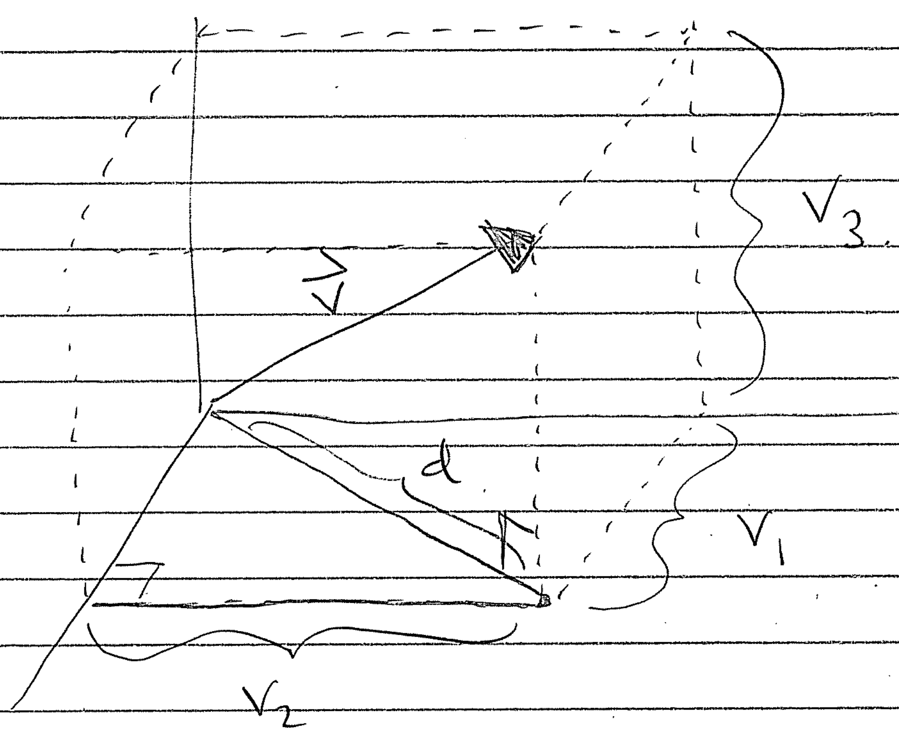
Pythagoras says:

$$\|\vec{v}\|^2 = v_1^2 + v_2^2$$

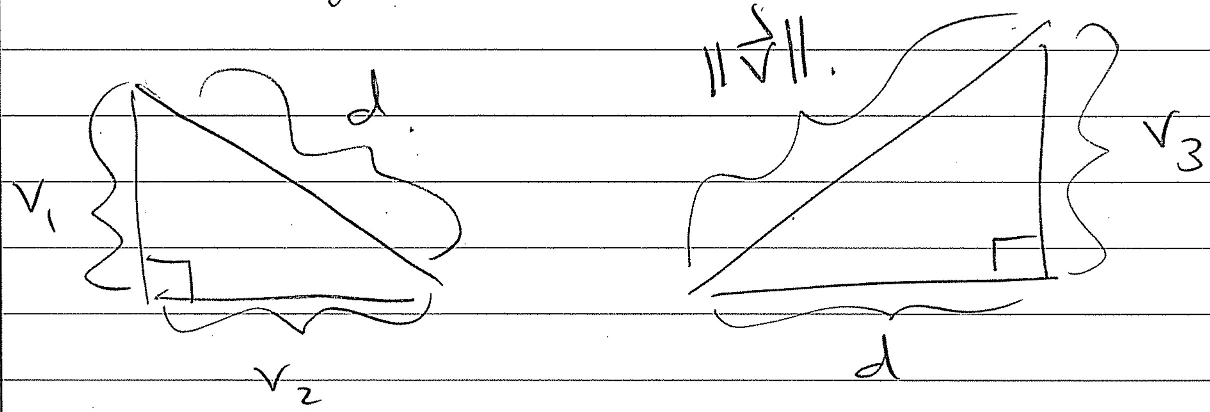
$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}.$$

What about in 3D? Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$$\|\vec{v}\| = ?$$



Two triangles



Pythagoras says:

$$d^2 = v_1^2 + v_2^2 \qquad \|\vec{v}\|^2 = d^2 + v_3^2$$

$$\begin{aligned}\|\vec{v}\|^2 &= d^2 + v_3^2 \\ &= v_1^2 + v_2^2 + v_3^2\end{aligned}$$

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Question: If $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$,

is it true that

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2} \quad ??$$

Answer: Sure, why not?

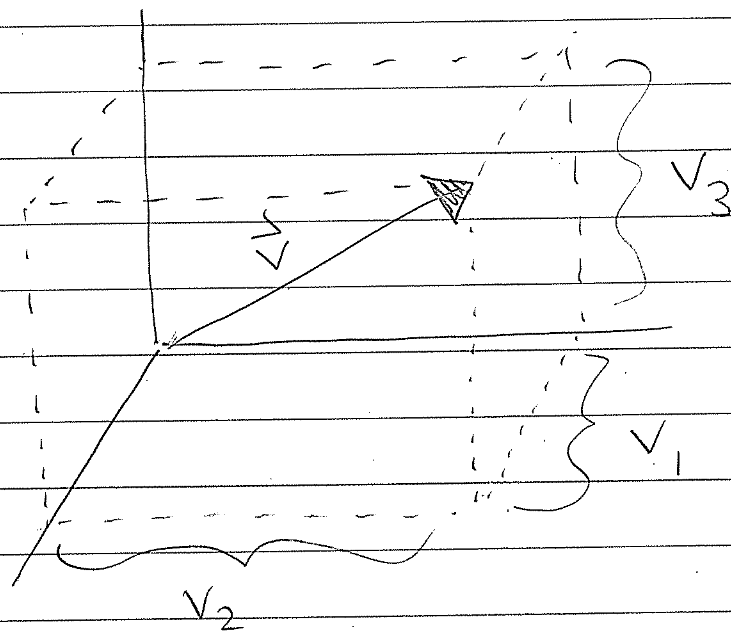
Wed Jan 16

Recall :

A vector is an ordered list of numbers

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

OR an arrow in space. (Descartes)



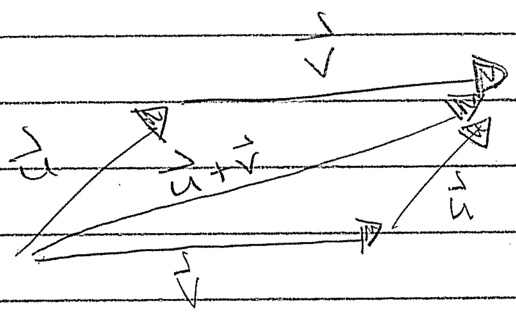
Vectors can be added.

If $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ then

$$\vec{u} + \vec{v} := \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix}$$

definition

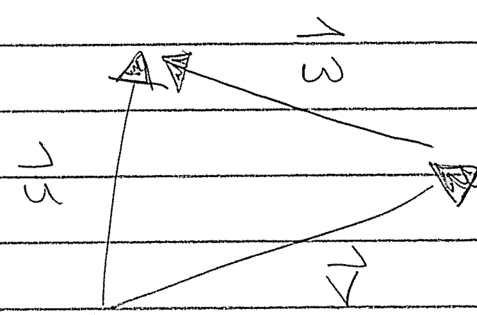
Geometrically, arrows are added head-to-tail:



$\vec{u}+\vec{v} = \vec{v}+\vec{u}$

Q: Can we <u>subtract</u> vectors?

Consider:

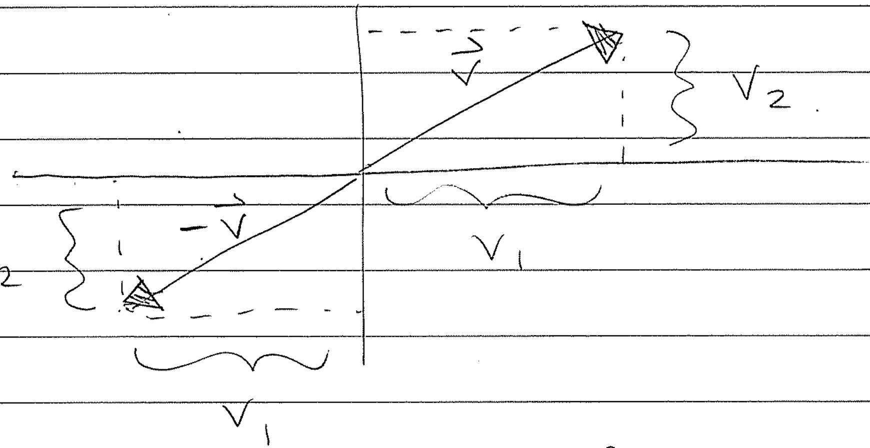


$\vec{v} + \vec{w} = \vec{u}$

We would like to say $\vec{w} =$ " $\vec{u} - \vec{v}$ "

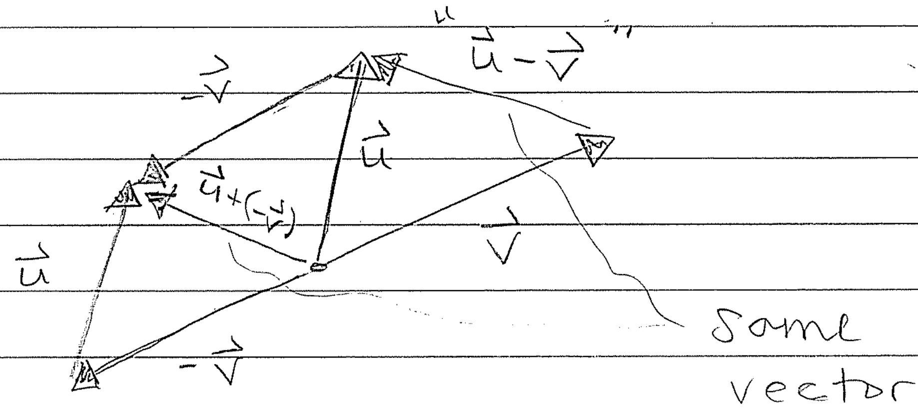
Does this make sense? Yes!

Given arrow $\vec{v}$, let $-\vec{v}$ be the same arrow with <u>opposite</u> <u>direction</u>.



We write $-\vec{v} = \begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix}$

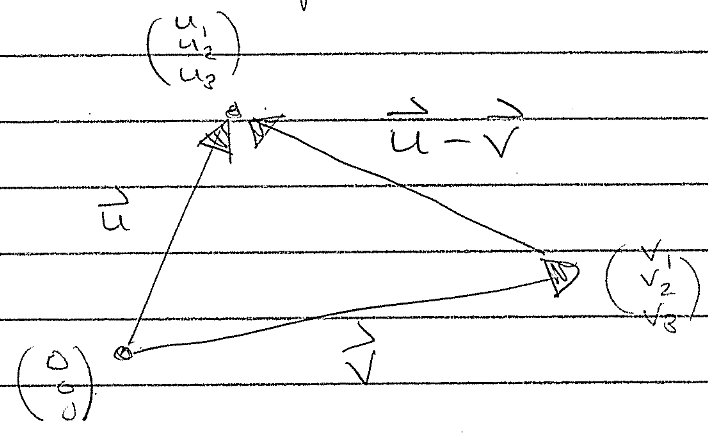
Then observe:



By definition we have

$$\vec{u}-\vec{v} := \vec{u}+(-\vec{v})$$

This allows us to compute the <u>distance</u> between two points.



The distance $\text{dist}(\vec{u},\vec{v})$ is the length of the arrow

$$\vec{u}-\vec{v} = \begin{pmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_3 - v_3 \end{pmatrix}$$

Recall Pythagoras:

$$\text{dist}(\vec{u},\vec{v})^2 = \|\vec{u}-\vec{v}\|^2$$

$$= (u_1-v_1)^2 + (u_2-v_2)^2 + (u_3-v_3)^2.$$

Keep Going ...

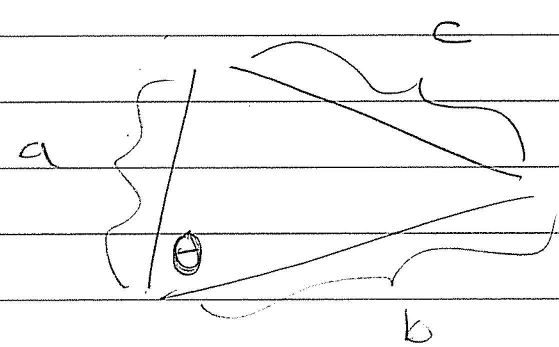
$$= (u_1^2 - 2u_1v_1 + v_1^2) + (u_2^2 - 2u_2v_2 + v_2^2) + (u_3^2 - 2u_3v_3 + v_3^2)$$

$$= (u_1^2 + u_2^2 + u_3)^2 + (v_1^2 + v_2^2 + v_3^2)$$

$$- 2(u_1v_1 + u_2v_2 + u_3v_3).$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\underbrace{(u_1v_1 + u_2v_2 + u_3v_3)}_{\text{what is this ?}}.$$

Think about a general triangle.



Ancient "Law of Cosines" says

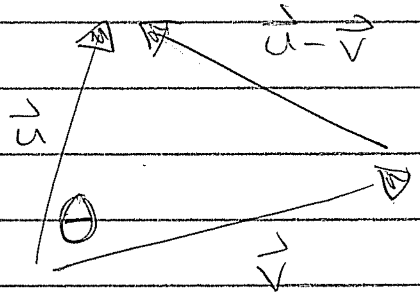
$$c^2 = a^2 + b^2 - \text{something}$$

$$= a^2 + b^2 - 2ab\cos\theta.$$

When $\cos\theta = 0$ (i.e. $\theta = 90°$) this is the Pythagorean Theorem.

$$c^2 = a^2 + b^2.$$

Now recall our triangle of vectors



We computed

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\left(u_1v_1 + u_2v_2 + u_3v_3\right).$$

Law of cosines says

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\,\|\vec{v}\|\cos\theta.$$

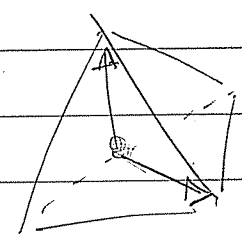
Conclusion: we have

$$u_1 v_1 + u_2 v_2 + u_3 v_3 = \|\vec{u}\| \, \|\vec{v}\| \cos \theta .$$

This is very useful!

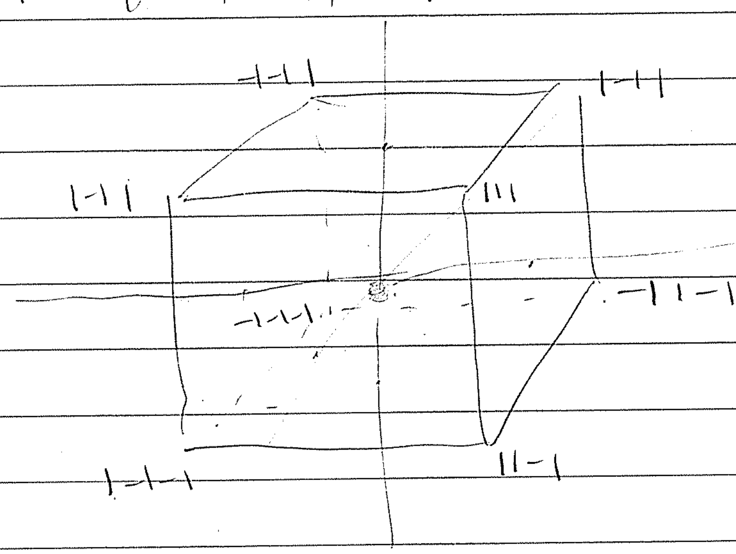
Application: The tetrahedral angle.

Consider a regular tetrahedron

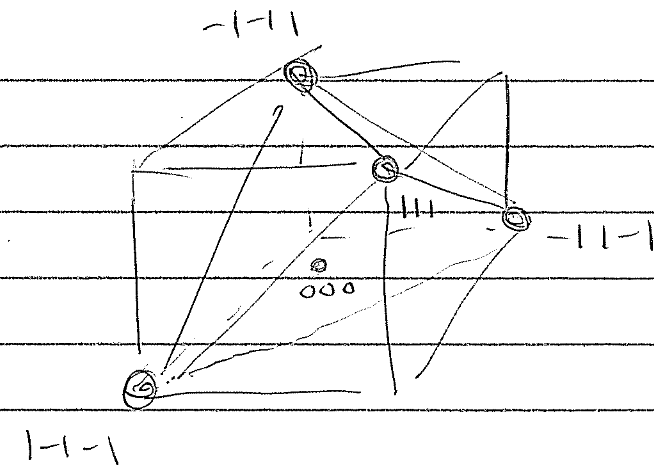


Compute the angle between two vertices.

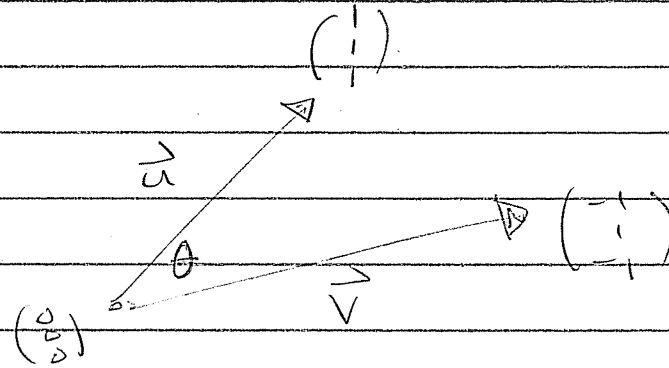
Answer: Write it in Cartesian coords.



cube has 8 vertices.



Let



Compute: $\cos\theta = \dfrac{u_1v_1 + u_2v_2 + u_3v_3}{\|\vec{u}\| \cdot \|\vec{v}\|}$

$= \dfrac{-1 + 1 + -1}{\sqrt{1^2+1^2+1^2} \cdot \sqrt{(-1)^2+1^2+(-1)^2}} = \dfrac{-1}{\sqrt{3}\cdot\sqrt{3}}$

$= -\dfrac{1}{3}$

Hence $\theta = \cos^{-1}\left(-\dfrac{1}{3}\right) \approx 109.47^\circ$

//

Fri Jan 18

HW 1 due Fri Jan 25 IN CLASS

Office Hours: Mon 2-3
Wed 3-4
and by appointment.

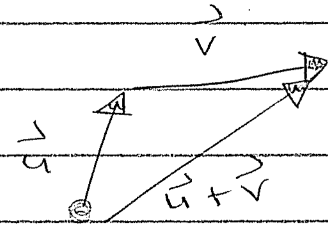
Webpage is up!

www.math.miami.edu/~armstrong/210sp13.html

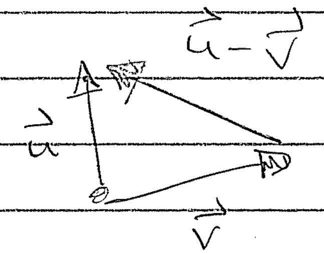
Recall: We can add and subtract vectors.

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\vec{u} + \vec{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix}$$



$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v}) = \begin{pmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_3 - v_3 \end{pmatrix}$$

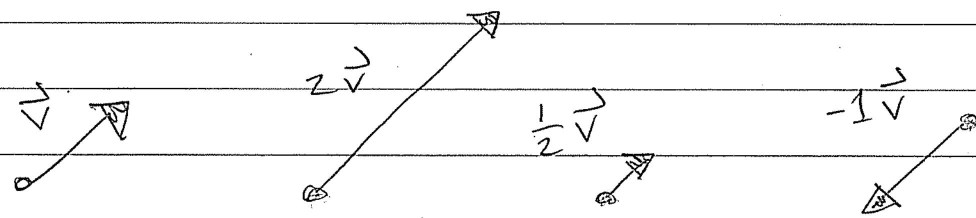


we can also <u>scale</u> vectors :

for any <u>number</u> α we define

$$\alpha \vec{v} = \alpha \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} := \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ \alpha v_3 \end{pmatrix}$$

(same direction, different length)



Q : Can we "multiply" vectors ?

Hmm . . . we could just say

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} := \begin{pmatrix} u_1 v_1 \\ u_2 v_2 \\ u_3 v_3 \end{pmatrix}$$

Turns out this is
NOT INTERESTING / USEFUL

Other ideas?

We got a hint last time when we found

$$\underbrace{u_1 v_1 + u_2 v_2 + u_3 v_3}_{} = \|\vec{u}\| \cdot \|\vec{v}\| \cos\theta$$

kind of like "multiplying" $\vec{u}$ and $\vec{v}$

In general, we define the "dot product" (or "inner product") of vectors

$\vec{u} = (u_1, u_2, \ldots, u_n)$ & $\vec{v} = (v_1, v_2, \ldots, v_n)$

by $\boxed{\vec{u} \cdot \vec{v} := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n}$

Strange Thing!

$\vec{u} \cdot \vec{v}$ is a <u>number</u>, not a <u>vector</u>

But otherwise it acts just like multiplication should e.g.

(1) $\vec{u} \circ \vec{v} = \vec{v} \circ \vec{u}$.

(2) $\vec{u} \circ (\vec{v} + \vec{w}) = \vec{u} \circ \vec{v} + \vec{u} \circ \vec{w}$

And it's a really <u>useful</u> notation:

$$\vec{v} \circ \vec{v} = v_1 v_1 + v_2 v_2 + \cdots + v_n v_n$$

$$= v_1^2 + v_2^2 + \cdots + v_n^2 = \|\vec{v}\|^2.$$

$$\Longrightarrow \|\vec{v}\| = \sqrt{\vec{v} \circ \vec{v}}$$

We can compute distances:

$$\text{dist}(\vec{u}, \vec{v})^2 = \|\vec{u} - \vec{v}\|^2$$

$$= (\vec{u} - \vec{v}) \circ (\vec{u} - \vec{v}).$$

$$= (\vec{u} - \vec{v}) \circ \vec{u} - (\vec{u} - \vec{v}) \circ \vec{v}.$$

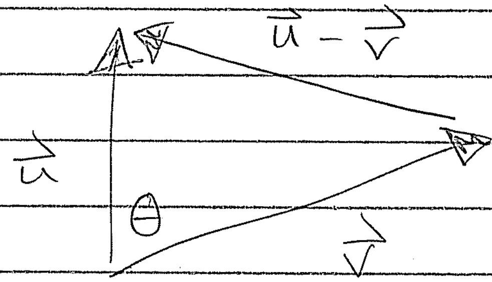
$$= \vec{u} \circ \vec{u} - \vec{v} \circ \vec{u} - [\vec{u} \circ \vec{v} - \vec{v} \circ \vec{v}]$$

$$= \vec{u} \circ \vec{u} + \vec{v} \circ \vec{v} - \vec{v} \circ \vec{u} - \vec{u} \circ \vec{v}.$$

$$= \vec{u} \circ \vec{u} + \vec{v} \circ \vec{v} - 2\vec{u} \circ \vec{v}$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u} \circ \vec{v}$$

Comparing with the triangle,



we get.

$$\vec{u} \circ \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta.$$

$$\cos\theta = \frac{\vec{u} \circ \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}.$$

$$\text{angle}(\vec{u}, \vec{v}) = \cos^{-1}\left(\frac{\vec{u} \circ \vec{v}}{\sqrt{\vec{u} \circ \vec{u}}\,\sqrt{\vec{v} \circ \vec{v}}}\right)$$

"All of geometry can be expressed in terms of dot products"

Q: But can we "really" multiply vectors?

vector $\times$ vector $=$ vector

A: Only in very special cases.

Example: In 2D we can define

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} := \begin{pmatrix} u_1 v_1 - u_2 v_2 \\ u_1 v_2 + u_2 v_1 \end{pmatrix}$$

This product has

amazing / useful / beautiful

properties!

Think: $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ "$=$" $u_1 + i u_2$

where $i = \sqrt{-1}$

Then we have

$$(u_1 + i u_2)(v_1 + i v_2)$$

$$= u_1 v_1 + i u_1 v_2 + i u_2 v_1 + \overset{-1}{\cancel{i^2}} u_2 v_2$$

$$= (u_1 v_1 - u_2 v_2) + i\,(u_1 v_1 + u_2 v_2).$$

There is no analogue in 3D

However, we <u>can</u> do it in

2D, 4D, 8D, 16D, etc...

Complex numbers

Quaternions
W.R. Hamilton
(1843)

...