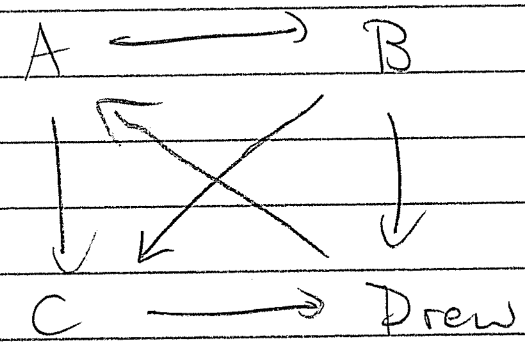
Mon Apr 22

Final Exam: Mon May 6 11:00 – 1:30

This week: Summary & Review

But first ... Google.

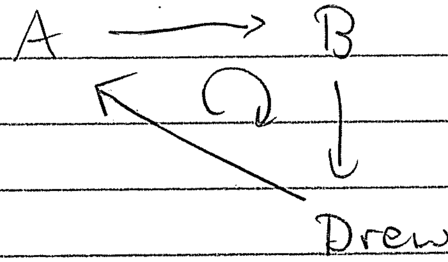
This weekend I hosted a round-robin ping-pong tournament. Here are the results:



"A → B" means A defeated B.

Q: Who won?
Rank the players fairly.

We have an immediate problem



There is a cycle. So any ranking will have an upset

Eg. $A > B > \text{Drew}$

But I defeated A, so I am upset.

Fact: Unfortunately, the probability of cycles goes to 100% as the # players $\to \infty$.

So we have to deal with upsets.

Idea: Try to minimize the upsets.

Eg.

$A > B > C > D$

Here there is only one upset.
But it's a big one.

☆ Here's the Google idea (based on the Perron - Frobenius theorem):
1907 1912

It should be worth more to defeat a better player.

Possible objection: How do we know who's better if we haven't ranked them yet??

The solution: Recursion!

Give everyone an initial score of 1.

$$\vec{V}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{matrix} A \\ B \\ C \\ D. \end{matrix}$$

Then for all $n \geq 0$ define the nth score vector by

$$\vec{V}_{n+1} = A \vec{V}_n \,,$$ where

$$A = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$ is the transition matrix.

Example:

$$\vec{V}_1 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix} \begin{array}{c} A \\ B \\ C \\ D. \end{array}$$

$$\vec{V}_2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ & 0 & 1 & 1 \\ & & 0 & 1 \\ 1 & & & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \end{pmatrix} \begin{array}{c} A \\ B \\ C \\ D. \end{array}$$

$$\vec{V}_3 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ & 0 & 1 & 1 \\ & & 0 & 1 \\ 1 & & & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 2 \\ 3 \end{pmatrix} \begin{array}{c} A \\ B \\ C \\ D. \end{array}$$

In words, the $(n+1)$th score of player P = the sum of the nth scores of the players that P defeated.

Example: continue...

$$\vec{v}_5 = \begin{pmatrix} 8 \\ 6 \\ 3 \\ 5 \end{pmatrix}, \quad \vec{v}_{10} = \begin{pmatrix} 35 \\ 34 \\ 21 \\ 26 \end{pmatrix}, \quad \vec{v}_{100} = \begin{pmatrix} 1037 \\ 933 \\ 547 \\ 731 \end{pmatrix}$$

It looks like $A > B >$ Drew $> C$ is the correct ranking.

How can we know for sure?
We must analyze

$$\vec{v}_n = A^n \vec{v}_0 \quad \text{as } n \to \infty$$

with eigen-analysis.

Perron-Frobenius says there exists a largest positive eigenvalue λ_{PF} for the matrix A.

such that the rescaled system

$$\vec{V}_n = \left(\frac{1}{\lambda_{PF}} A\right)^n \vec{V}_0$$

converges to an equilibrium. $\vec{V}_\infty$.
After that we have

$$\vec{V}_\infty = \vec{V}_{\infty+1} = \frac{1}{\lambda_{PF}} A \vec{V}_\infty$$

$$\Longrightarrow \quad A\vec{V}_\infty = \lambda_{PF} \vec{V}_\infty.$$

The equilibrium is a λ_{PF}-eigenvector!

In our case, my computer says

$$\lambda_{PF} = 1.395336994, \ldots$$

and the equilibrium is

$$\vec{V}_\infty = \begin{array}{cc} \begin{pmatrix} 0.321\ldots \\ 0.283\ldots \\ 0.165\ldots \\ 0.230\ldots \end{pmatrix} & \begin{array}{l} \text{A} \\ \text{B} \\ \text{C} \\ \text{Drew.} \end{array} \end{array}$$

These are the correct scores :)

Wed Apr 24

Exam 2 out of 30.

Average 22
Median 23.5
St. Dev. 5.7

Appoximate (!) Grade Ranges:

A = 26 — 30
B = 18 — 25
C = 13 — 17
D = 7 — 12

Final Exam Mon May 6
11:00 — 1:30, here.

Today & Friday:

Summary and Review.

Q: What is linear algebra?

The central object of linear algebra is a linear equation in n unknowns

(*) $$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b,$$

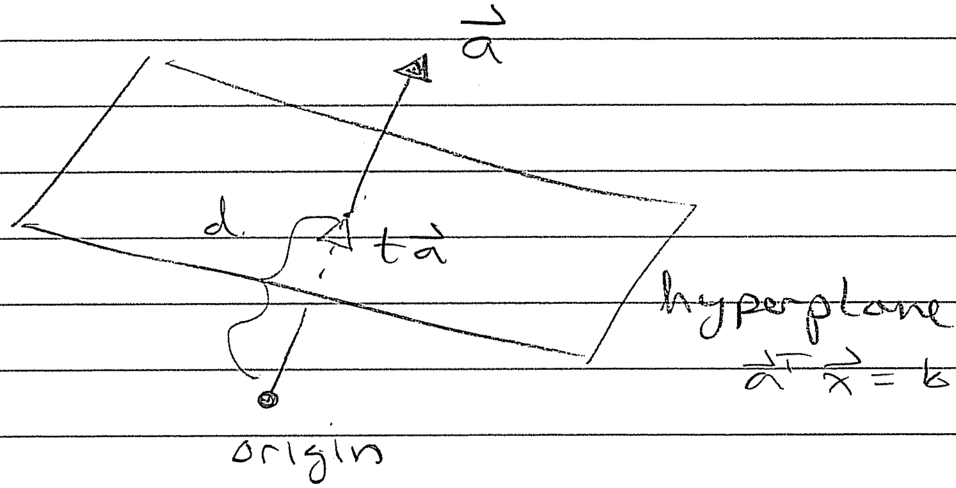
which can be thought of as an $(n-1)$-dim "hyperplane" in n-dim space $\mathbb{R}^n$.

Rephrase (*) in terms of dot product

$$(a_1\ a_2\ \cdots\ a_n)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b.$$

$$\vec{a}^T \vec{x} = b$$

Picture:



Compute the distance d. Which vector $t\vec{a}$ is on the hyperplane? We have

$$\vec{a}^T(t\vec{a}) = b.$$

$$t\,\vec{a}^T\vec{a} = b$$

$$t = b/\vec{a}^T\vec{a} = b/\|\vec{a}\|^2.$$

Then $d = t\|\vec{a}\|$

$$= \frac{b}{\|\vec{a}\|^2}\|\vec{a}\| = \frac{b}{\|\vec{a}\|}. \qquad \blacksquare$$

The central problem of linear algebra is to solve a system of m linear equations in n unknowns:

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

In short: $A\vec{x} = \vec{b}$

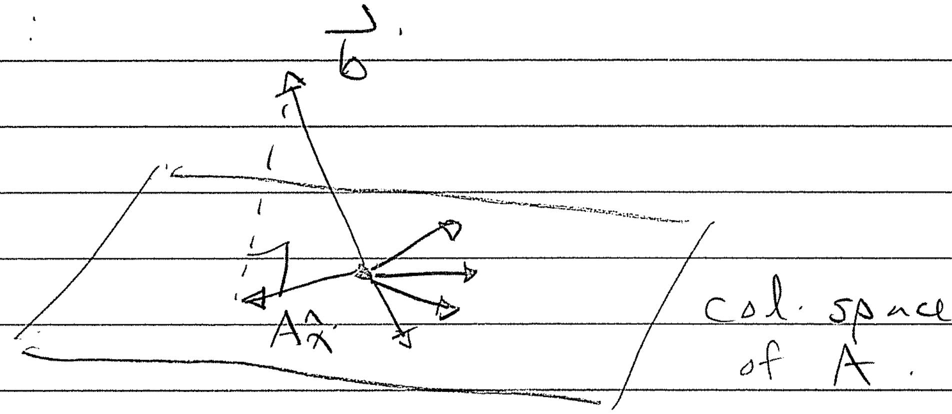
Row Picture: Find the geometric intersection of m hyperplanes in $\mathbb{R}^n$.

Column Picture: Solve.

$$x_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = \vec{b}.$$

That is, say whether $\vec{b}$ is in the column space of A or not, and if so, where?

<u>Picture</u>:



If $\vec{b}$ is outside the col space, solve

$$A^T A \hat{x} = A^T \vec{b}$$

instead.

<u>How</u> to Solve? Gaussian Elimination

Step 1 : Down Elimination

- Find a nonzero pivot on the top left.
- Eliminate entries below the pivot.
- Repeat on the smaller system

$$\begin{pmatrix} * & * & * & \cdots & * \\ 0 & & & & \\ 0 & & \text{Repeat.} & & \\ 0 & & & & \end{pmatrix}$$

Step 2 : Up Elimination.
(Same as Back - Substitution)

- Eliminate entries above the bottom - right pivot.
- Repeat.

Result:

$$\begin{pmatrix} * & * & 0 & * & 0 & * \\ 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

RREF.

Read the solution from the RREF.

Example:

$$\begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$$

Pivot variables : x_1, x_3

Free variables : x_2, x_4.

Let $x_2 = s$, $x_3 = t$. Solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 - 3s - 2t \\ s \\ 2 + t \\ t \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

This is a (2-dimensional) plane in 4-dimensional space $\mathbb{R}^4$.

Remarks:

① With m equations and n unknowns, the solution is <u>probably</u>

$(n-m)$-dimensional.

[and empty if $m > n$.].

② If $\vec{u}, \vec{v}$ are any two solutions

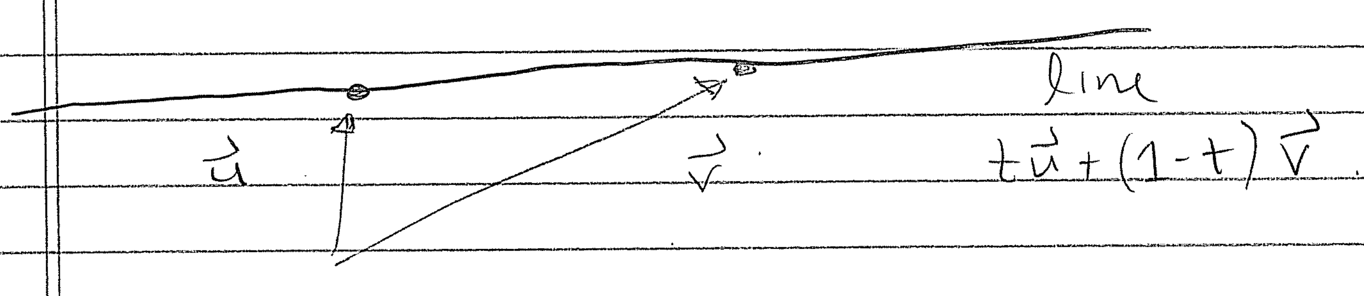
$$A\vec{u} = \vec{b} \quad \text{and} \quad A\vec{v} = \vec{b}.$$

then we get a whole line of solutions

$$t\vec{u} + (1-t)\vec{v} \quad \text{for all } t.$$

In words:

The solutions form a "flat" thing



Fri Apr 26

Final Exam: Mon May 6, 11:00 – 1:30, <u>here</u>.

Today: Summary & Review.

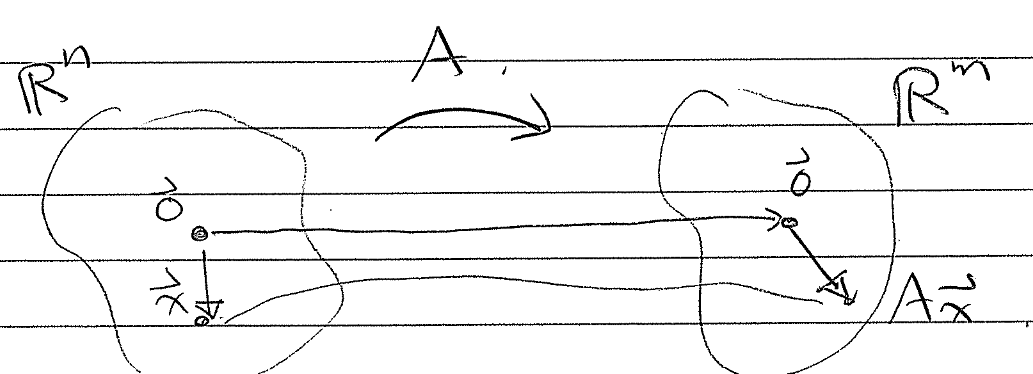
Last time we discussed $A\vec{x} = \vec{b}$ as a system of linear equations.

Today, A is a <u>function</u>.

If A has shape $m \times n$,

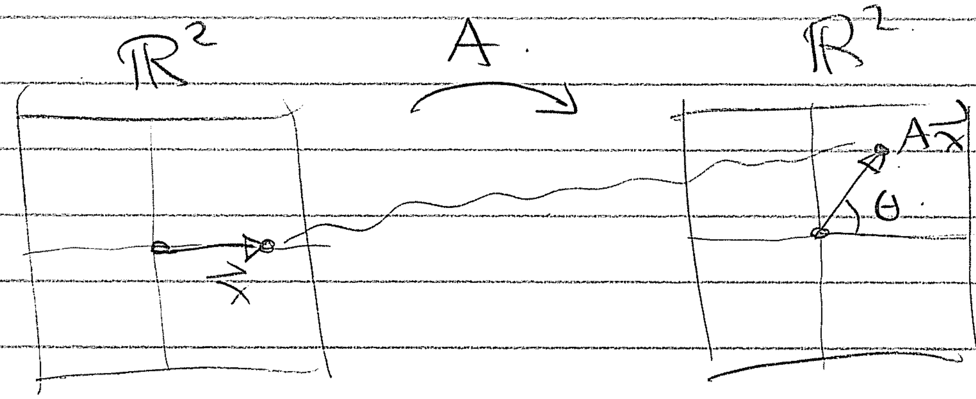
$$m\left\{\underbrace{\begin{pmatrix} & A & \end{pmatrix}}_{n}\right. \left.\begin{pmatrix}\vec{x}\end{pmatrix}\right\}n = \left.\begin{pmatrix}\vec{b}\end{pmatrix}\right\}m$$

Then the rule $\vec{x} \mapsto A\vec{x}$ is a <u>function</u> from $\mathbb{R}^n$ to $\mathbb{R}^m$



Example: The matrix $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

rotates vectors in $\mathbb{R}^2$ by angle θ c.c.w.



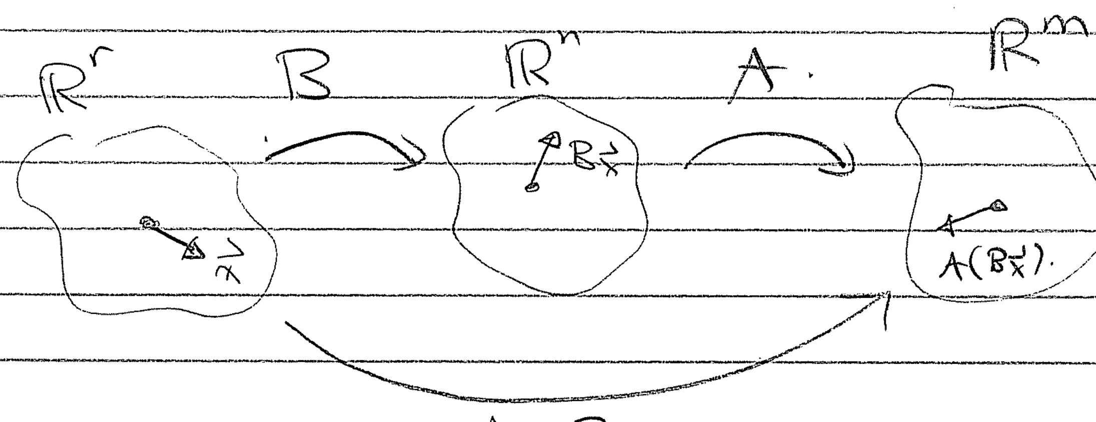
Q: Who cares?

A: Functions can be composed.!

Given matrices A shape $m \times n$

B shape $n \times r$

we have functions



$A \circ B$.

(do B and then do A).

Definition: Let AB be the $m \times r$ matrix that represents the function $A \circ B$.

That is, we have

$$A(B\vec{x}) = (AB)\vec{x}.$$

Q: But what <u>is</u> the matrix AB?

A: i,j entry AB = (ith row A)(jth col B)

ith row AB = (ith row A) B.

jth col AB = A (jth col B)

A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is called <u>invertible</u> if there is another function $g: \mathbb{R}^m \to \mathbb{R}^n$ such that

$f \circ g$ = "do nothing" function $\mathbb{R}^m \to \mathbb{R}^m$
$g \circ f$ = "do nothing" function $\mathbb{R}^n \to \mathbb{R}^n$.

Then we write $g = f^{-1}$.

The $m \times n$ matrix A is called invertible if there exists $n \times m$ matrix B such that

$$AB = I \quad (m \times m)$$
$$BA = I \quad (n \times n).$$

It turns out this can only happen when $m = n$ (rectangles are never invertible)

To compute the inverse (Gauss-Jordan):

$$\left(A \mid I \right) \xrightarrow{\text{RREF}} \left(I \mid A^{-1} \right)$$

If a row of zeros appears, then A is NOT invertible.

Why: • the rows of A were not independent

Note: A is invertible $\iff$ A^T is invertible

and if so, $(A^T)^{-1} = (A^{-1})^T$.

Proof: If A^{-1} exists, then

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I.$$

Hence $(A^T)^{-1}$ exists and $= (A^{-1})^T$ $\blacksquare$

Then: A is NOT invertible

$$\Updownarrow$$

A^T is NOT invertible

$$\Updownarrow$$

rows of A^T have a relation (Gauss-Jordan fails)

$$\Updownarrow$$

columns of A have a relation

$$\Updownarrow$$

$A\vec{x} = \vec{0}$ for some $\vec{x} \neq \vec{0}$

$$\Updownarrow$$

0 is an eigenvalue of A

In general we say λ is an eigenvalue of A if there exists $\vec{x} \neq \vec{0}$ with

$$A\vec{x} = \lambda\vec{x}$$
$$A\vec{x} = \lambda I\vec{x}$$
$$A\vec{x} - \lambda I\vec{x} = \vec{0}$$
$$(A - \lambda I)\vec{x} = \vec{0}.$$

Thus: λ is an eigenvalue of A.

$$\Updownarrow$$

0 is an eigenvalue of $A-\lambda I$

$$\Updownarrow$$

$A-\lambda I$ is NOT invertible.

$$\Updownarrow$$

"$\det(A-\lambda I)=0$."

We can be more explicit for $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then $\det\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$

$$= \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$= (a-\lambda)(d-\lambda) - bc.$$

The characteristic equation is

$$(a-\lambda)(d-\lambda) - bc = 0$$
$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0.$$

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

Q: Who cares?

A: If you are going to apply the function A repeatedly, i.e.,

$$\vec{v}_n = A\vec{v}_{n+1}$$

Then you should:

1. Find the eigenvalues
2. Find the eigenvectors
3. Hope you have lots of eigenvectors
4. Write everything in terms of eigenvectors
5. Your life is easy.