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This is a closed book test. No electronic devices are allowed. If two students submit exams in which any solution has been copied, **both students will receive a score of zero**. There are 7 pages and 4 problems, worth a total of 30 points.

Problem 1. [5 points] We wish to **project** the vector $\vec{b}$ onto the line spanned by $\vec{a}$. The answer is $t\vec{a}$ for some number t:
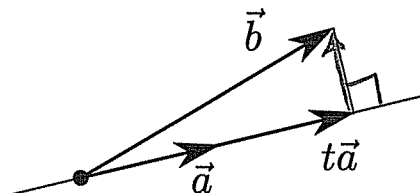
(a) Write a true **equation** involving $\vec{a}$, $\vec{b}$, and t. [Hint: Dot product.]

$$\vec{a}^T(\vec{b} - t\vec{a}) = 0$$

(b) **Solve** your equation to find t.

$$\vec{a}^T\vec{b} - t\,\vec{a}^T\vec{a} = 0$$

$$t\,\vec{a}^T\vec{a} = \vec{a}^T\vec{b}$$

$$t = \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}$$

(c) Tell me the **matrix** P such that $P\vec{b} = t\vec{a}$ gives the projection.

$$P\vec{b} = \left(\frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}\right)\vec{a} = \vec{a}\left(\frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}\right)$$

$$= \left(\frac{\vec{a}\,\vec{a}^T}{\vec{a}^T\vec{a}}\right)\vec{b} \implies P = \frac{\vec{a}\,\vec{a}^T}{\vec{a}^T\vec{a}}$$

Problem 2. [6 points] Consider three data points $\begin{pmatrix} t \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. We wish to find the **best-fit line** of the form $C + Dt = b$.

(a) Express the three equations $C + D(-1) = 0$, $C + D(0) = 1$, $C + D(1) = 3$ as a **single matrix equation** $A\vec{x} = \vec{b}$ (which, unfortunately, has no solution).

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

(b) Write down the **normal equation** $A^T A \hat{x} = A^T \vec{b}$ (which does have a solution).

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$
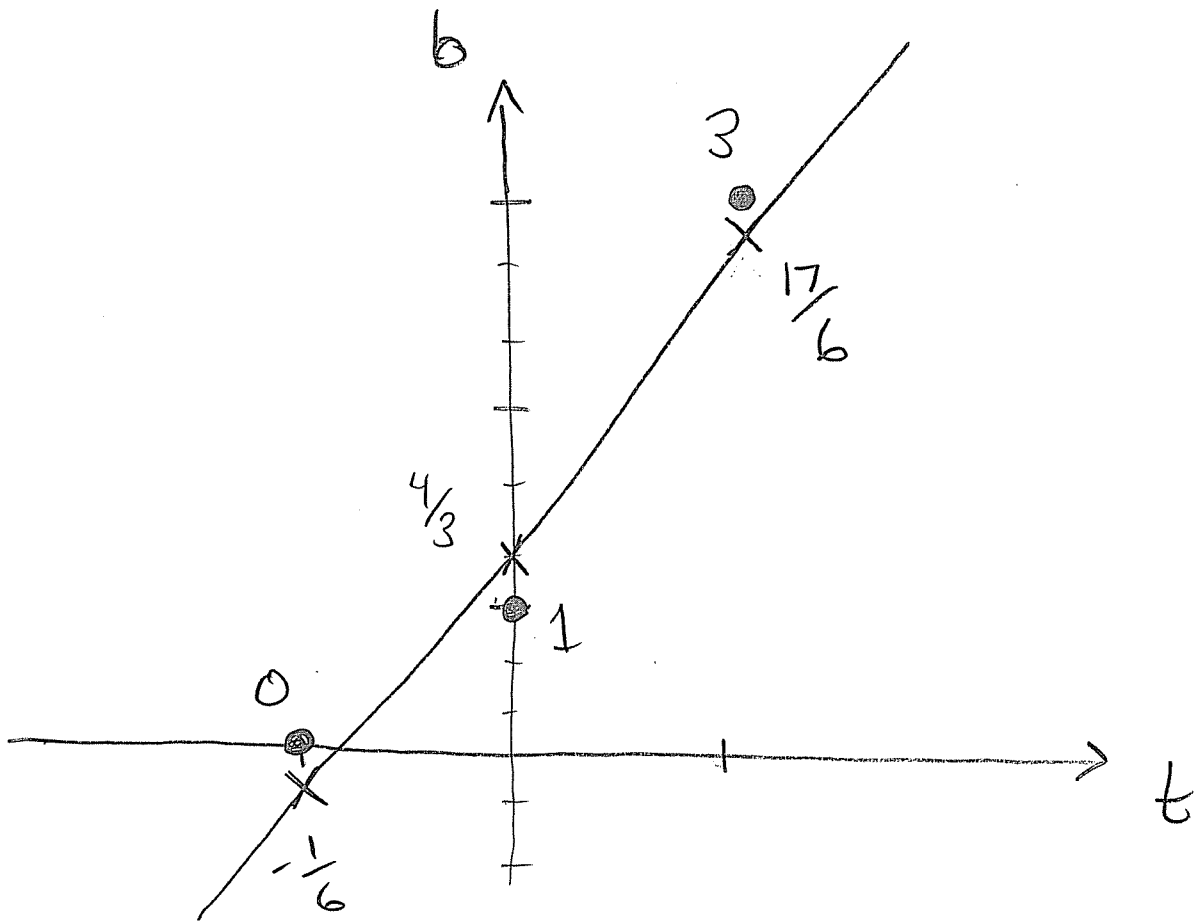
$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

(c) **Solve** the normal equation to find C and D.

$$\begin{cases} 3C + 0D = 4 \\ 0C + 2D = 3 \end{cases} \implies \begin{array}{l} C = 4/3 \\ D = 3/2. \end{array}$$

Best Fit line: $\dfrac{4}{3} + \dfrac{3}{2} t = b$

(d) **Draw** the data points and the best-fit line.

Problem 3. [9 points] We wish to solve the linear recurrence $\vec{v}_{n+1} = A\vec{v}_n$, with matrix

$$A = \begin{pmatrix} .2 & .8 \\ .4 & .6 \end{pmatrix}$$

and **initial condition** $\vec{v}_0 = (3,0)$.

(a) I will tell you that the eigenvalues of A are 1 and $-.2$. Compute the eigen**vectors**.

$$\lambda = 1 \qquad \begin{array}{cc} .2-1 & .8 \\ .4 & .6-1 \end{array} \left| \begin{array}{c} 0 \\ 0 \end{array} \right. \longrightarrow \begin{array}{cc} x & y \\ -.8 & .8 \\ \sim\!\sim\!\sim & \end{array} \left| \begin{array}{c} 0 \\ 0 \end{array} \right.$$

$$\Longrightarrow \quad \text{The line} \quad \begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = -.2 \qquad \begin{array}{cc} .2+.2 & .8 \\ .4 & .6+.2 \end{array} \left| \begin{array}{c} 0 \\ 0 \end{array} \right. \longrightarrow \begin{array}{cc} x & y \\ .4 & .8 \\ \sim\!\sim\!\sim & \end{array} \left| \begin{array}{c} 0 \\ 0 \end{array} \right.$$

$$\Longrightarrow \quad \text{The line} \quad \begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

(b) Express $\vec{v}_0 = (3,0)$ in terms of eigenvectors.

We have

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

(c) Use your answer from part (b) to find an **explicit formula** for $\vec{v}_n$.

$$\vec{v}_n = A^n \begin{pmatrix} 3 \\ 0 \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + A^n \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-.2)^n \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + (-1)^n 2/5^n \\ 1 - (-1)^n 1/5^n \end{pmatrix}.$$

(d) Draw the **phase portrait** of the system, and draw your trajectory starting from $\vec{v}_0 = (3,0)$.
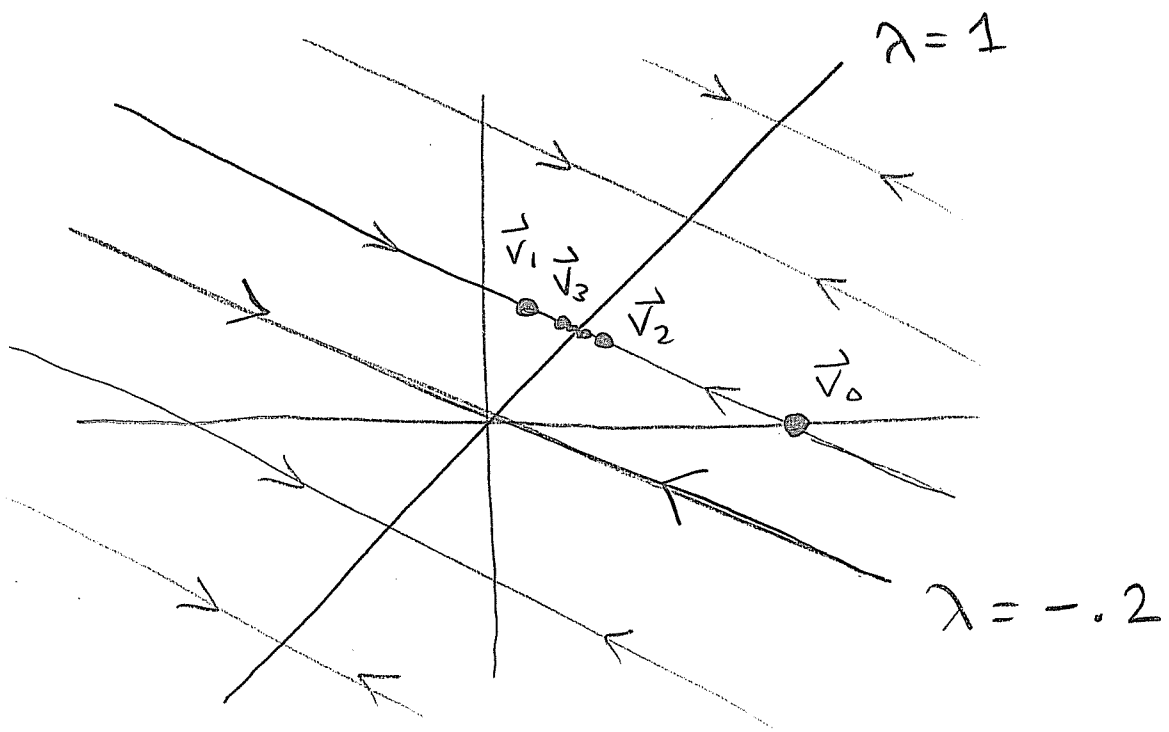
(e) Tell me the limit of $\vec{v}_n$ as $n \to \infty$

$$\vec{v}_n \longrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{as} \quad n \longrightarrow \infty$$

Problem 4. [**10 points**] Let A be any matrix with independent columns.

(a) Tell me the matrix P that **projects** orthogonally onto the column space of A.

$$P = A(A^TA)^{-1}A^T$$

(b) What are the eigenvalues of P? [Hint: One of them is 1.]

0 and 1.

(c) What are the eigenvalues of $I - P$?

$1 - 0$ and $1 - 1$

1 and 0

(d) What are the eigenvalues of $2P - I$?

$2 \cdot 0 - 1$ and $2 \cdot 1 - 1$

1 and -1.

(e) We know that the matrix from part (a) satisfies $P^2 = P$ (you **don't** need to show this). Use this fact to show that $(2P - I)^2 = I$.

$$(2P-I)(2P-I) = 4P^2 - 2PI - 2IP + I^2$$

$$= 4P - 2P - 2P + I$$

$$= I$$

(f) Is the matrix $2P - I$ invertible? If so, tell me its inverse.

Yes. $(2P-I)(2P-I) = I$

$\implies (2P-I)^{-1} = 2P-I$

(g) [**1 bonus point**] Describe the function $2P - I$ geometrically.

It performs a reflection across the column space of A.

Example: $\vec{b}$

$(2P-I)\vec{b}$