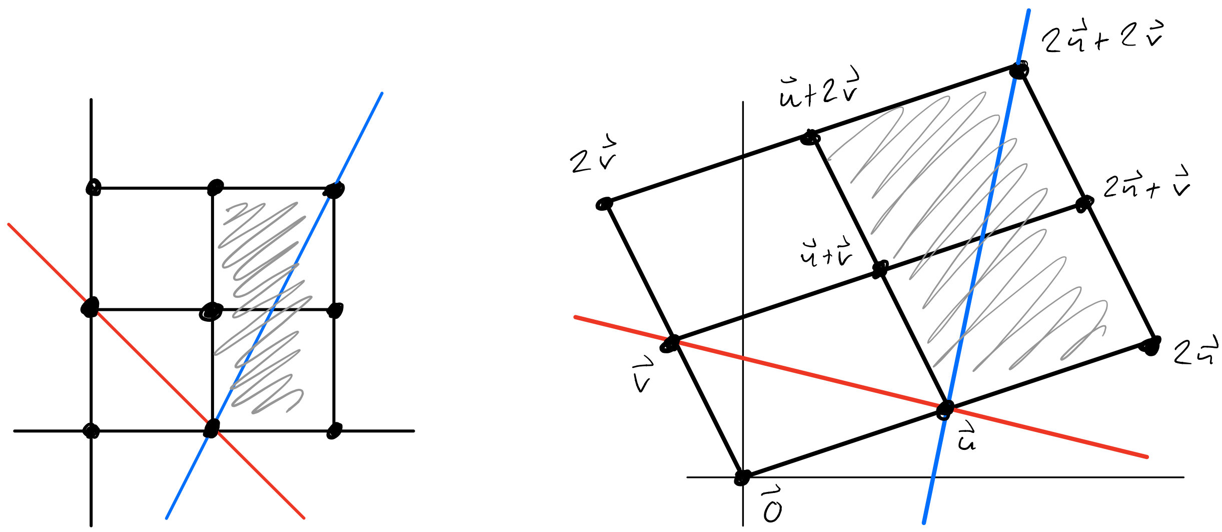
Today: HW1 Discussion (& more?)

Tues: Quiz 1 for 20 minutes at the beginning of class.

Problem 1:



Idea: Point (x,y) on the left corresponds to point

$$x\vec{u}+y\vec{v} = x\begin{pmatrix}3\\1\end{pmatrix}+y\begin{pmatrix}-1\\2\end{pmatrix}$$

on the right. Observe it is "the same picture" in two different "coordinate systems."

Later we will write this as

$$(x,y) \mapsto \underbrace{x\begin{pmatrix}3\\1\end{pmatrix} + y\begin{pmatrix}-1\\2\end{pmatrix}}_{\text{``}\begin{pmatrix}3&-1\\1&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\text{''}}$$

and we will call this
"matrix multiplication"

In other words, the "matrix"

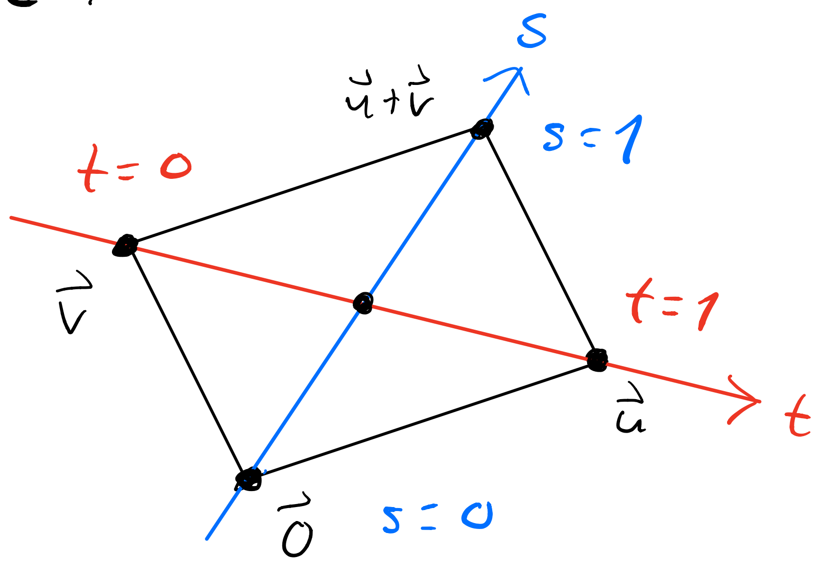
$$\begin{pmatrix}3&-1\\1&2\end{pmatrix}$$

transforms the left picture into the right picture.

Problem 2(a):

Given two points $\vec{u}$ & $\vec{v}$ living in n-dimensional space, we say that $(\vec{u}+\vec{v})/2$ is the "midpoint" of $\vec{u}$ & $\vec{v}$.

Picture:



Recall, points $\vec{0}, \vec{u}, \vec{v}, \vec{u}+\vec{v}$ are vertices of a parallelogram.

Consider the diagonal lines:

red : $t\vec{u}+(1-t)\vec{v}$

blue : $s(\vec{u}+\vec{v})$

Where do they intersect?

$$t\vec{u}+(1-t)\vec{v} = s(\vec{u}+\vec{v})$$

$$t\vec{u}+(1-t)\vec{v} = s\vec{u}+s\vec{v}.$$

I claim that the coefficients of $\vec{u}$ & $\vec{v}$ must be equal, so

$$\begin{cases} t = s \,, \\ 1-t = s \,. \end{cases}$$

This system of 2 equations has the unique solution $t = 1/2$ & $s = 1/2$.

[Indeed: $t = s = 1-t$

$t = 1-t$

$2t = 1$

$t = 1/2$.]

Conclusion: The intersection point is $\frac{1}{2}\vec{u} + \frac{1}{2}\vec{v} = (\vec{u}+\vec{v})/2$.

More generally, suppose we have particle of mass m_1 at position $\vec{u}$ & particle of mass m_2 at position $\vec{v}$.

Archimedes' Law of the Lever:

Then, the center of mass is the point

$$\frac{m_1 \vec{u} + m_2 \vec{v}}{m_1 + m_2} = \left(\frac{m_1}{m_1+m_2}\right)\vec{u} + \left(\frac{m_2}{m_1+m_2}\right)\vec{v}.$$

Previous example was $m_1 = m_2$.

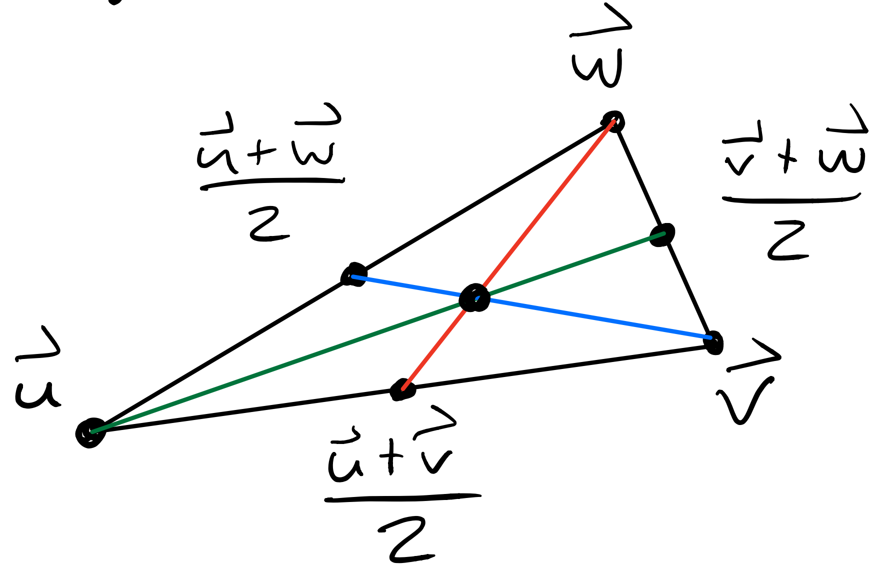
Problem 2(b):

Given points $\vec{u}, \vec{v}, \vec{w}$ living in n-dimensional space, what is the significance of the point

$$\frac{\vec{u} + \vec{v} + \vec{w}}{3} \quad ?$$

Claim: This is the "centroid" of the triangle $\vec{u}, \vec{v}, \vec{w}$.

Meaning:



The centroid is defined as the intersection of the lines that connect each vertex to midpoint of opposite side.

Proof that $\frac{\vec{u}+\vec{v}+\vec{w}}{3}$ = centroid:

red line: $t\left(\frac{\vec{u}+\vec{v}}{2}\right)+(1-t)\vec{w}$

plug in parameter $t=2/3$ to get

$$\frac{2}{3}\left(\frac{\vec{u}+\vec{v}}{2}\right)+\left(1-\frac{2}{3}\right)\vec{w}=\frac{\vec{u}+\vec{v}+\vec{w}}{3}.$$

Similar computation shows this point is on all 3 lines.

More generally, if we have particles of mass m_1, m_2, m_3 at the points $\vec{u}, \vec{v}, \vec{w}$, then the center of mass is

$$\frac{m_1 \vec{u} + m_2 \vec{v} + m_3 \vec{w}}{m_1 + m_2 + m_3}$$

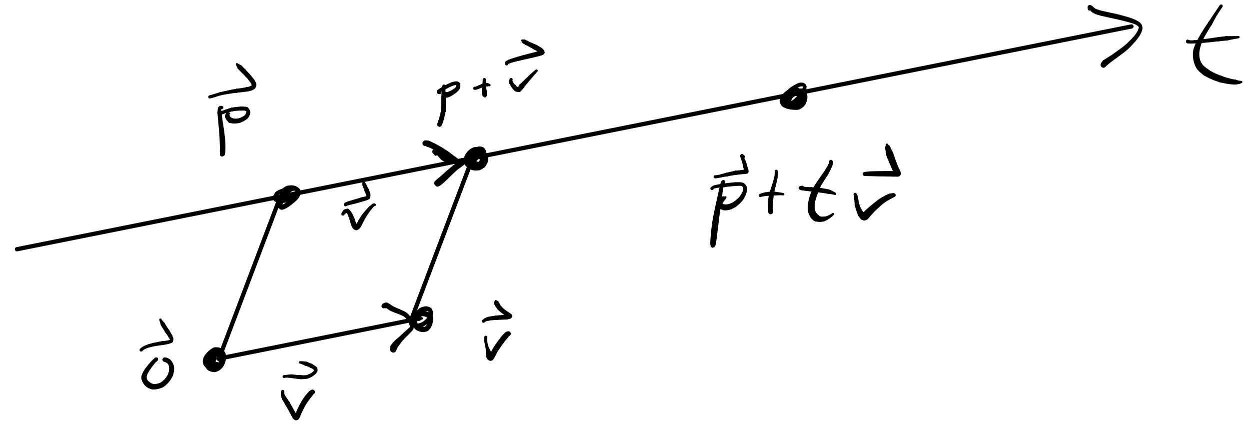
The centroid is the center of mass for 3 equal masses.

Problem 3: See the solutions.

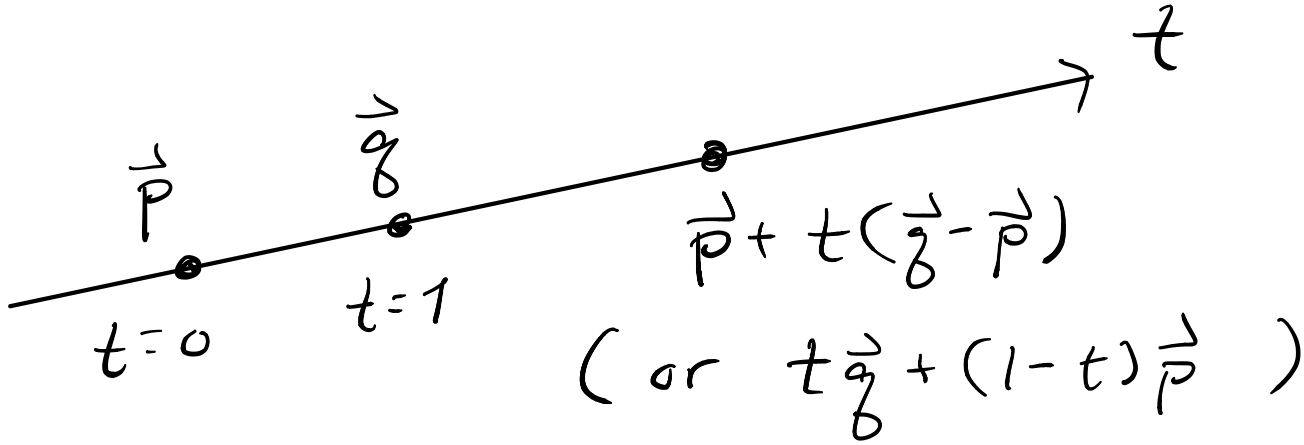
Problems 4 & 5: Lines in 2D.

Recall: A line in $\mathbb{R}^2$ can be described by

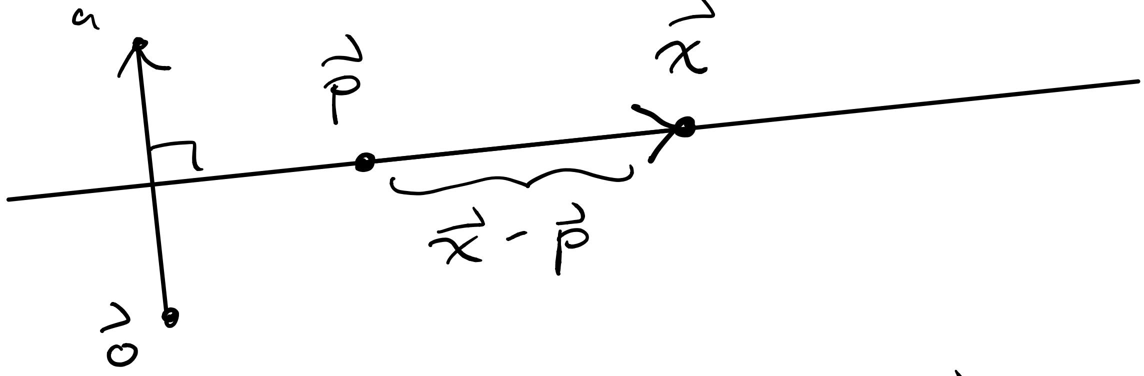
- a point $\vec{p}$ & direction vector $\vec{v}$:



- two points $\vec{p}$ & $\vec{q}$:



- a point $\vec{p}$ & normal vector $\vec{a}$:



To be on the line, point $\vec{x}$ must satisfy

$$\vec{a}\circ(\vec{x}-\vec{p}) = 0.$$

$$\vec{a}\circ\vec{x} - \vec{a}\circ\vec{p} = 0$$

$$\vec{a}\circ\vec{x} = \vec{a}\circ\vec{p}$$

vector   scalar.

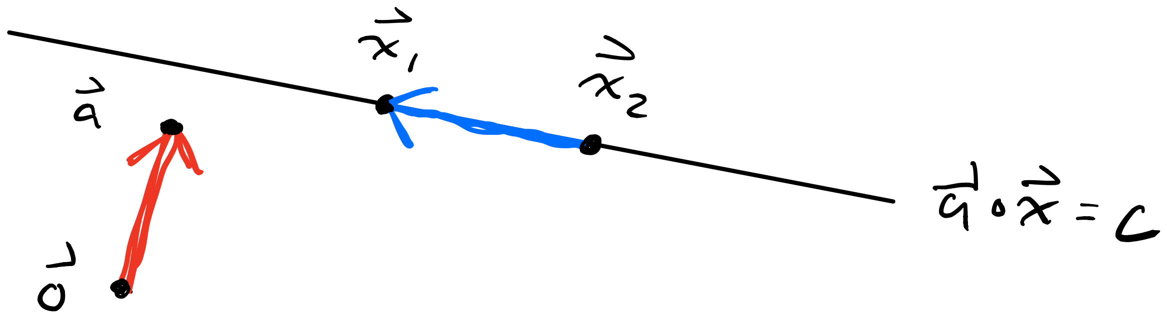
Conversely (5b), if two points $\vec{x}_1$ & $\vec{x}_1$ satisfy the equation $\vec{a}\circ\vec{x} = c$, i.e., if

$$\vec{a}\circ\vec{x}_1 = c \quad \& \quad \vec{a}\circ\vec{x}_2 = c$$

then we have

$$\vec{a}\circ(\vec{x}_1-\vec{x}_2) = \vec{a}\circ\vec{x}_1 - a\circ\vec{x}_2$$
$$= c - c$$
$$= 0.$$

Meaning: vectors $\vec{a}$ & $\vec{x}_1-\vec{x}_2$ are <u>perpendicular</u>.



The red & blue arrows are perpendicular!

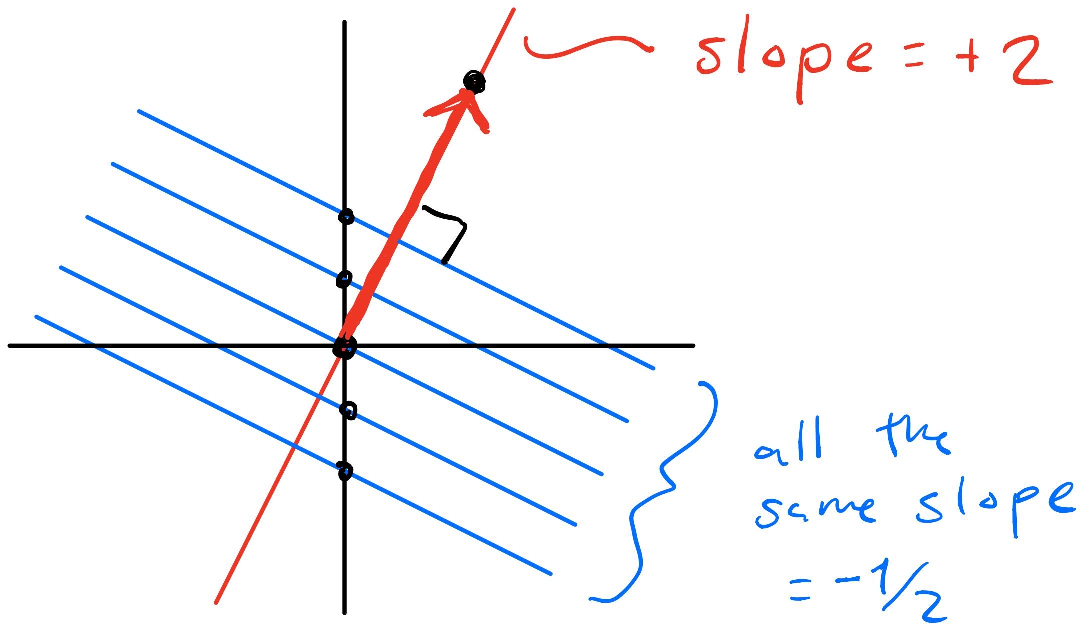
Conclusion: The lines

$\vec{a}\circ\vec{x} = c$ & $t\vec{a} = t(a,b)$

$ax+by = c$

are perpendicular to each other.

Picture from 5(a):



The blue lines

$$\begin{pmatrix}1\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix} = c$$

$$x + 2y = c$$

$$y = -x/2 + c/2$$

are $\perp$ to the red line

$$t\begin{pmatrix}1\\2\end{pmatrix}$$

Next: What happens in 3D?

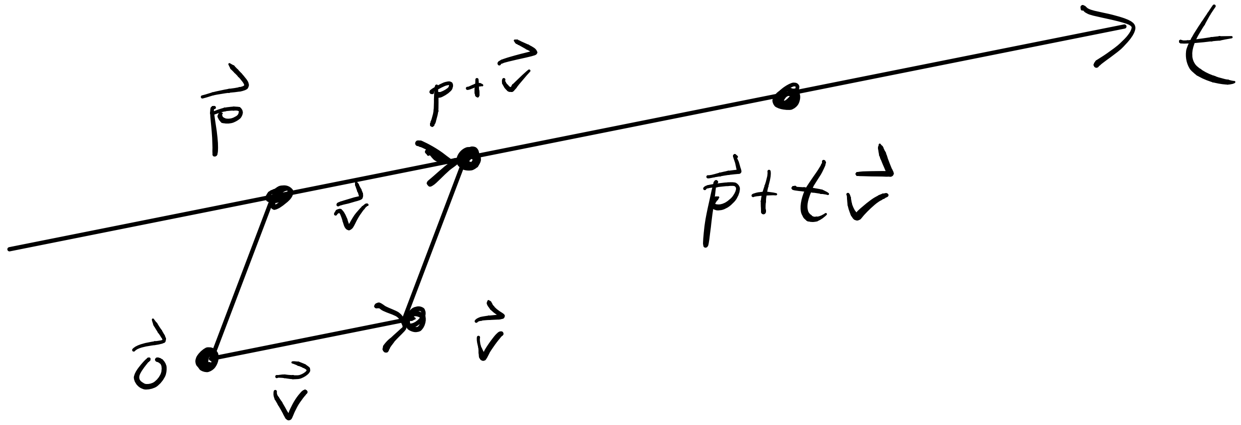
What is the equation of a line in 3D space $\mathbb{R}^3$?

TRICK QUESTION !

There is no such thing. A line in $\mathbb{R}^3$ cannot be described with a single equation !

So how can we describe a line ?

The (point, direction vector) and (point, point) descriptions of a line work in any number of dimensions.



e.g. this picture lives in $\mathbb{R}^{100}$.

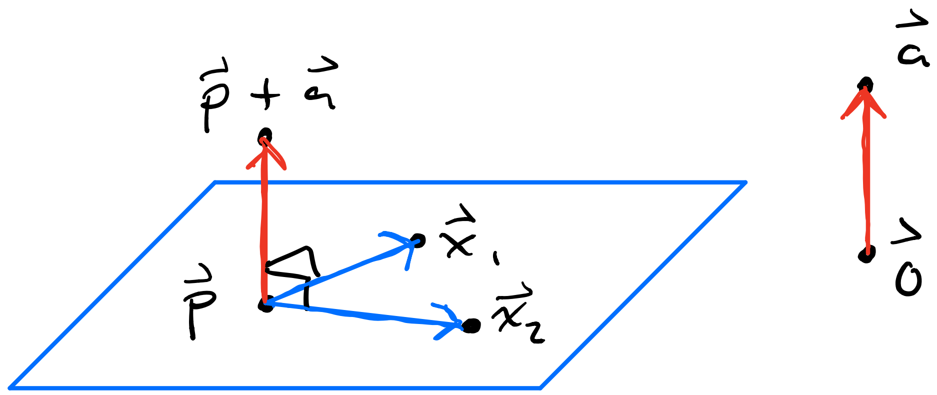
But the (point, normal vector) equation does not describe a line in general.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = d$$

$$ax + by + cz = d.$$

What shape does this describe?

It's a plane. Picture:



Any point $\vec{x}$ on the plane must have vector $\vec{x}-\vec{p} \perp$ vector $\vec{a}$, which means

$$\vec{a} \cdot (\vec{x} - \vec{p}) = 0$$

$$\vec{a} \cdot \vec{x} - \vec{a} \cdot \vec{p} = 0$$

$$\vec{a} \cdot \vec{x} = \vec{a} \cdot \vec{p}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \text{some constant } d$$

$$\boxed{ax + by + cz = d}$$

Conclusion: This equation defines a <u>plane</u> in $\mathbb{R}^3$ that is $\perp$ to the vector $\vec{a} = (a, b, c)$.

For next week:

The general "linear equation"

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

defines an "$(n-1)$-dimensional hyperplane" living in n-dimensional space. Never mind for now!

For the quiz:

See examples on the Summer 2020 webpage. There will be 2 problems, so 10 minutes each.

Topics:

- Arithmetic of vectors.
  - computations
  - pictures
  - $\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| \cos\theta$.

- Lines in the plane
  - convert between various forms

e.g. $t\vec{u} + (1-t)\vec{v}$ vs. $ax + by = c$.
parametrization   normal equation

- Planes & Hyperplanes are NOT on Quiz 1.