HW 4 due now.

Quiz 4: Tuesday, beginning of class.

Today: HW4 Discussion.

Problem 1: For any $m \times n$ matrix A, the function $\mathbb{R}^n \to \mathbb{R}^m$ defined by $\vec{x} \mapsto A\vec{x}$ is "linear," i.e. sends parallelograms to parallelograms.

Proof: Let $A = \begin{pmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{pmatrix}$ where $\vec{a}_1, \ldots, \vec{a}_n \in \mathbb{R}^m$ are the "column vectors."

For any $\vec{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$ we define

$$A\vec{u} := u_1\vec{a}_1 + u_2\vec{a}_2 + \cdots + u_n\vec{a}_n.$$

If $\vec{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$, $s, t \in \mathbb{R}$, then we have

$$s\vec{u} + t\vec{v} = (su_1 + tv_1, \ldots, su_n + tv_n)$$

and hence

$$A(s\vec{u}+t\vec{v}) = (su_1+tv_1)\vec{a}_1 + \cdots + (su_n+tv_n)\vec{a}_n.$$

On the other hand,

$$s A\vec{u} + t A\vec{v}$$

$$= s(u_1\vec{a}_1 + \cdots + u_n\vec{a}_n) + t(v_1\vec{a}_1 + \cdots + v_n\vec{a}_n).$$

Are these two expressions the same?

Yes ✓

This fact is important because we (secretly) use it to compute the product of matrices.

Recall: let A be $k \times l$
B $\quad l \times m$
C $\quad m \times n$.

so that AB is $k \times m$
BC is $l \times n$.

The matrices $A(BC)$ & $(AB)C$ are both defined & have shape $k \times n$.

Claim: $A(BC) = (AB)C$.

"matrix multiplication is associative."

The proof of this is a <u>picture</u>:

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "C"] \arrow[rr, bend left, "BC"] \arrow[r, bend right, "C"'] & \mathbb{R}^m \arrow[r, "B"] \arrow[rr, bend right, "AB"'] & \mathbb{R}^l \arrow[r, "A"] \arrow[r, bend left, "A"] & \mathbb{R}^k
\end{tikzcd}$$

You get to the same place either way.

[we will see next week that the associative property is <u>surprisingly</u> useful.]

Example from Problem 2:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Compute product $\vec{y}^T A \vec{x}$ in two different ways:

- $A\vec{x} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

  $\vec{y}^T (A\vec{x}) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = 6$

- $\vec{y}^T A = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \end{pmatrix}$

  $(\vec{y}^T A)\vec{x} = \begin{pmatrix} 3 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 6$

SAME!

Problem 3: Special Matrices.

Find 2×2 matrices with special properties.

(a) $N \neq 0$, $N^2 = 0$.

Easiest example: $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Jargon: We say N is a "nilpotent" matrix if $N \neq 0$ but $N^m = 0$ for some m. General Picture:

$$N = \underbrace{\begin{pmatrix} 0 & & & * \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix}}_{m} \Big\} m$$

Then always have $N^m = 0$.

(b) $F \neq I$, $F^2 = I$.

Claim: $F_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ works.

Check:

$$F_\theta^2 = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

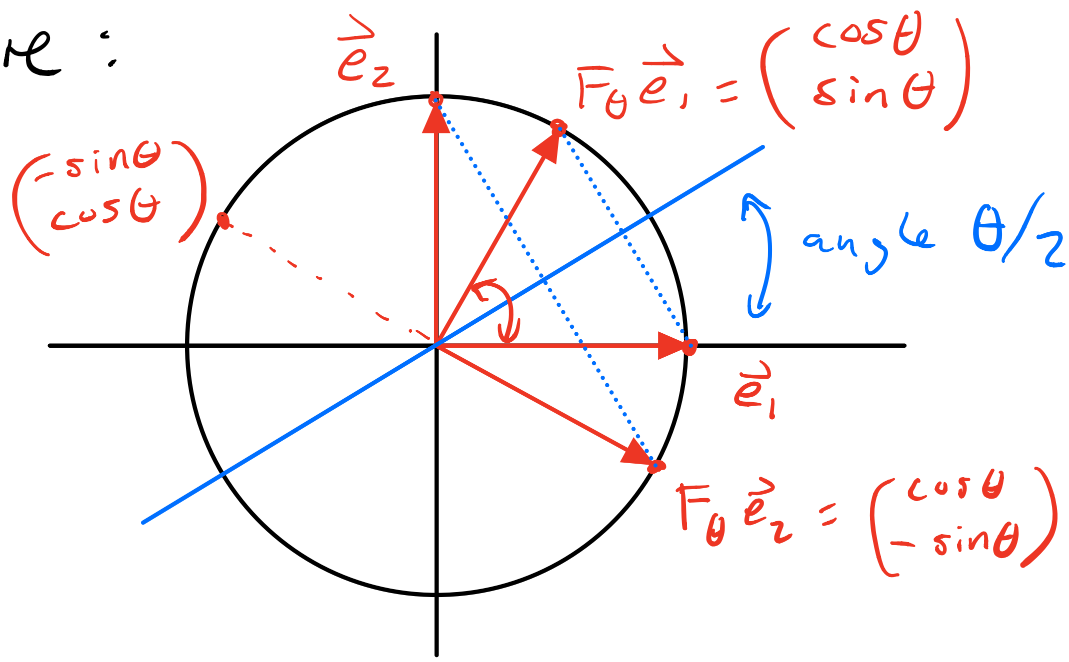
$$= \begin{pmatrix} \overbrace{\cos^2\theta + \sin^2\theta}^{1} & \overbrace{\cos\theta\sin\theta - \cos\theta\sin\theta}^{0} \\ \underbrace{\cos\theta\sin\theta - \cos\theta\sin\theta}_{0} & \underbrace{\sin^2\theta + \cos^2\theta}_{1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

Why did that happen?

Because F_θ is a <u>reflection</u> matrix.

Picture:



This is the <u>reflection</u> across the line with angle $\theta/2$. Hence

$$F_\theta^2 = \text{reflect twice}$$
$$= \text{do nothing} = I.$$

Examples:

$$F_{0^\circ} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \text{reflect across } x\text{-axis}.$$

$$F_{180^\circ} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \text{reflect across } y\text{-axis}.$$

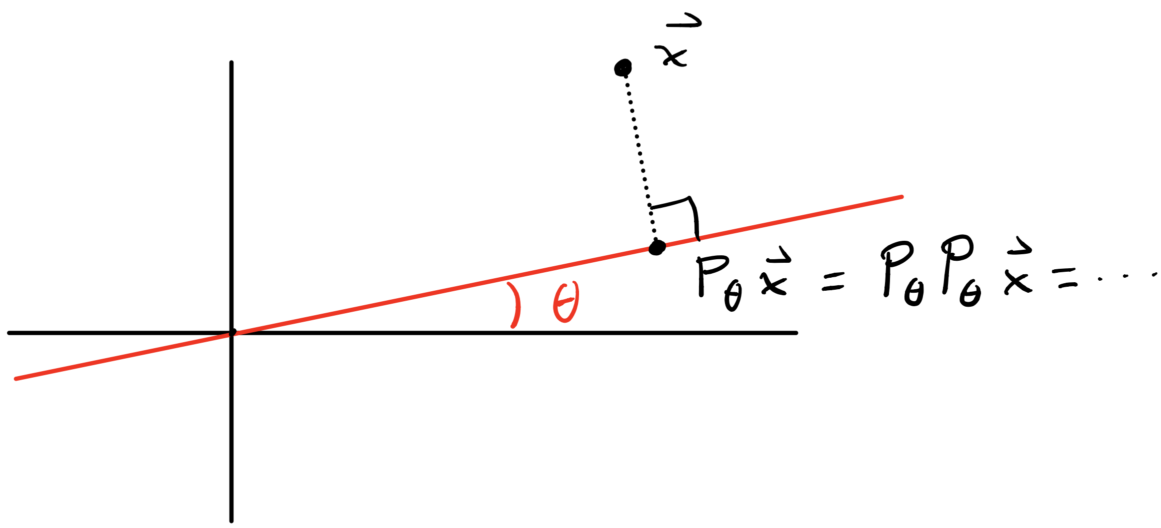
(c) $P \neq 0$, $P \neq I$, $P^2 = P$.

Claim: $$P_\theta = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$$

works.

Why? We will see next week that this is the matrix that projects onto the line of slope θ:



It's not 0 or I. And we must have

$$P_\theta^2 = P_\theta$$

Because projecting twice is the same as projecting once!

[Check if you want.]

(d) Rotation by $90°$.

Problem 4: Compute an inverse.

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 4 \end{pmatrix}$.

$$(A \mid I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$\overset{\text{RREF}}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -2 & 3 & -1 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) = (I \mid B)$$

Claim: $AB = BA = I$, so that

$B = A^{-1} =$ <u>the</u> inverse of A.

See solutions for a check.

What is it good for? Consider:

$$\begin{cases} x + y + z = a, \\ x + 2y + 2z = b, \\ x + 3y + 4z = c. \end{cases}$$

Write this as a matrix equation:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$A\vec{x} = \vec{b}$$

Multiply on the left by the inverse:

$$\cancel{A^{-1}A}\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a - b \\ -2a + 3b - c \\ a - 2b + c \end{pmatrix}$$

We just solved infinitely many systems of equations!

We "inverted the system."

Why does the method work?

$(A \mid I) \rightsquigarrow (I \mid A^{-1})$?

Let's look at a 2×2 matrix:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}.$$

Want to find $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

But recall:

$$A(j\text{th col } B) = j\text{th col } AB.$$

So $AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ~~becomes~~ two linear systems:

$$A(1\text{st col } B) = 1\text{st col } I$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A(2\text{nd col } B) = 2\text{nd col } I$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We can solve these two systems separately to compute $\begin{pmatrix} a \\ c \end{pmatrix}$ & $\begin{pmatrix} b \\ d \end{pmatrix}$.

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 2 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & -2 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right)$$

$$\rightsquigarrow \begin{cases} a + 0c = -1 \\ 0a + c = 1 \end{cases}$$

$$\rightsquigarrow \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 2 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -2 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & -\frac{1}{2} \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} \end{array}\right)$$

$$\rightsquigarrow \begin{cases} b + 0 = 1 \\ 0 + d = -\frac{1}{2} \end{cases}$$

$$\rightsquigarrow \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$$

We conclude that the inverse matrix is

$$A^{-1} = B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -\frac{1}{2} \end{pmatrix}$$

TRICK: We can save time by solving both systems simultaneously:

$$\left(\begin{array}{cc|c|c} 1 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right)$$

just don't draw these bars

$$(A \mid I) \rightsquigarrow (I \mid A^{-1})$$

Sometimes it goes wrong.

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix}$.

TRY to compute A^{-1}:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 3 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right)$$

oops!

There is no pivot in the 3rd row! This means that the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix}$ is

NOT INVERTIBLE!

But I claim that we could have predicted this without doing <u>any</u> computations.

Observe that the 2nd & 3rd columns of A are equal, which gives us a nontrivial <u>column relation</u>

$$0(\text{col } 1) + 1(\text{col } 2) - 1(\text{col } 3) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$0\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

But this can be factored:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus we have $A\vec{x} = \vec{0}$ for some nonzero vector $\vec{x} \neq \vec{0}$.

Theorem (Problem 5b): If $A\vec{x} = \vec{0}$ for some $\vec{x} \neq \vec{0}$ then A^{-1} does not exist.

Proof: Suppose that $A\vec{x} = \vec{0}$ and $\vec{x} \neq \vec{0}$. If A^{-1} were to exist then we would have

$$A\vec{x} = \vec{0}$$

$$\rightsquigarrow A^{-1}A\vec{x} = A^{-1}\vec{0}$$

$$\rightsquigarrow \vec{x} = \vec{0}$$

which is a <u>contradiction</u>.

Therefore A^{-1} does not exist. ///