HW6 is due now.

There is no Quiz 6.

There is no Final Exam.

The Final Project is due Tues. Dec 1.

Min 2 pages, Max ~10 pages.

Point form summary of what you learned.

Today: HW6 Discussion

& Grand Finale.

Problem 1: Important Formula.

If $A\vec{u} = \lambda\vec{u}$ then $A^n\vec{u} = \lambda^n\vec{u}$.

Proof "by induction":

It's true for $n=1$. ✓

Assume it's true for n, i.e., assume that $A^n\vec{u} = \lambda^n\vec{u}$. Then it's also true for $n+1$ because

$$A^{n+1}\vec{u} = (AA^n)\vec{u}$$

$$
\begin{aligned}
&= A(A^n\vec{u}) \\
&= A(\lambda^n\vec{u}) \\
&= \lambda^n(A\vec{u}) \\
&= \lambda^n(\lambda\vec{u}) \\
&= \lambda^{n+1}\vec{u} \quad \checkmark
\end{aligned}
$$

QED.

Consequence: If $A\vec{u} = \lambda\vec{u}$, $A\vec{v} = \mu\vec{v}$ then for any scalars a & b, have

$$
\begin{aligned}
&A^n(a\vec{u} + b\vec{v}) \\
&= aA^n\vec{u} + bA^n\vec{v} \\
&= a\lambda^n\vec{u} + b\mu^n\vec{v}.
\end{aligned}
$$

previous

More Generally: Consider a polynomial expression in one variable

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n.$$

If A is a square matrix we can "plug A into f" to get a matrix

$$f(A) := a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n.$$

Important Formula:

If $A\vec{u} = \lambda \vec{u}$ then $f(A)\vec{u} = f(\lambda)\vec{u}$.

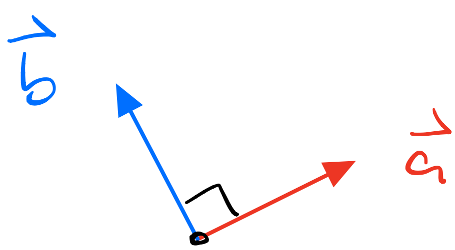
Proof:

$$\begin{aligned}
f(A)\vec{u} &= (a_0 I + a_1 A + \cdots + a_n A^n)\vec{u} \\
&= a_0 I\vec{u} + a_1 A\vec{u} + a_2 A^2\vec{u} + \cdots + a_n A^n \vec{u} \\
&= a_0 \vec{u} + a_1 \lambda \vec{u} + a_2 \lambda^2 \vec{u} + \cdots + a_n \lambda^n \vec{u} \\
&= (a_0 + a_1 \lambda + a_2 \lambda^2 + \cdots + a_n \lambda^n)\vec{u} \\
&= f(\lambda)\vec{u} \quad \checkmark
\end{aligned}$$

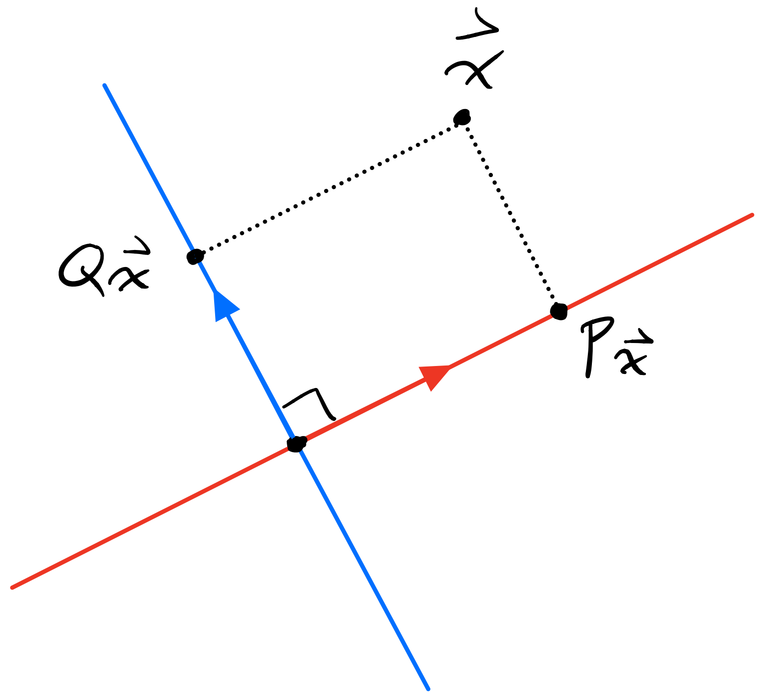
Problem 2: Consider $\vec{a}, \vec{b} \in \mathbb{R}^2$ with

$$\vec{a}\cdot\vec{a} = \vec{b}\cdot\vec{b} = 1 \ \& \ \vec{a}\cdot\vec{b} = 0.$$

Picture:



Let $P = \vec{a}\vec{a}^T$ & $Q = \vec{b}\vec{b}^T$ be the 2×2 matrices that project onto the lines $t\vec{a}$ & $t\vec{b}$, respectively.



Note that $\vec{a}$ & $\vec{b}$ are the eigenvectors of P & Q with

$$\begin{cases} P\vec{a} = 1\vec{a} \\ P\vec{b} = 0\vec{b} \end{cases} \quad \& \quad \begin{cases} Q\vec{a} = 0\vec{a} \\ Q\vec{b} = 1\vec{b} \end{cases}$$

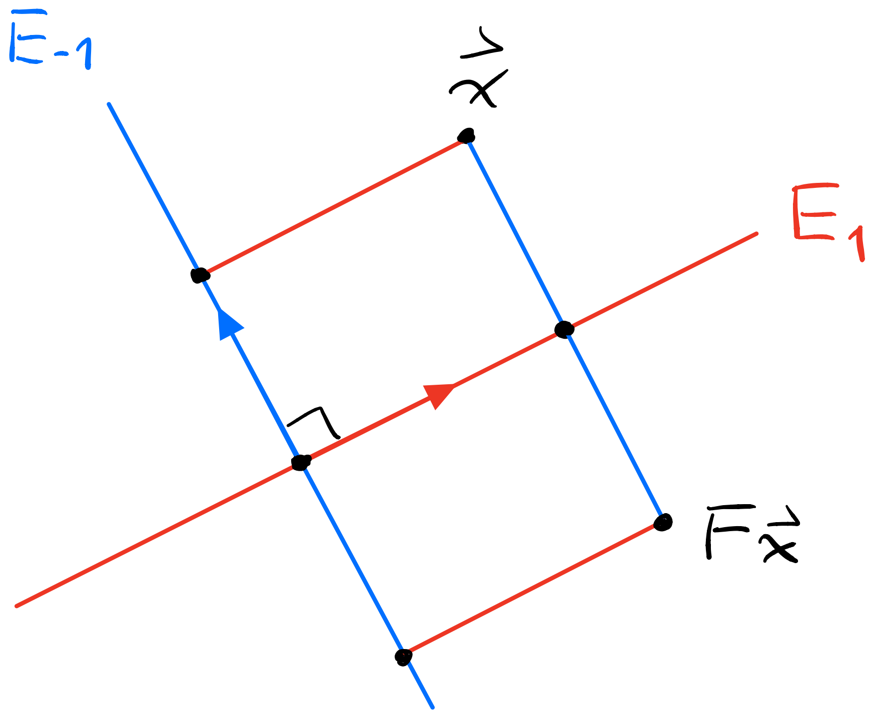
What is $F = 2P - I$?

I claim that F also has the same eigenvectors. What are the e.values?

$$F\vec{a} = (2P - I)\vec{a}$$
$$= 2P\vec{a} - \vec{a} = 2\vec{a} - \vec{a} = 1\vec{a}$$

$$F\vec{b} = (2P - I)\vec{b}$$
$$= 2P\vec{b} - \vec{b} = 2\vec{0} - \vec{b} = -1\vec{b}.$$

So here are the eigenspaces of F:



Apply F: red coord gets mult. by eigenvalue 1 & blue coord gets multiplied by eigenvalue -1.

Thus F reflects across "red axis."

Problem 3: Rotation Matrix.

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}.$$

Eigenvalues:

$$\det\begin{pmatrix} c-\lambda & -s \\ s & c-\lambda \end{pmatrix} = 0.$$

$$(c-\lambda)(c-\lambda) + s^2 = 0$$

$$\lambda^2 - 2c\lambda + \cancel{c^2 + s^2} = 0$$

$$\lambda^2 - 2c\lambda + 1 = 0.$$

$$\lambda = \left(2c \pm \sqrt{(-2c)^2 - 4}\right)/2$$

$$= \left(2c \pm \sqrt{4c^2-4}\right)/2$$

$$= c \pm \sqrt{c^2-1}$$

$$= c \pm \sqrt{-s^2}$$

$$= c \pm \sqrt{-1}\, s$$

$= \cos\theta \pm i\sin\theta$

When is this a real number?

If $\theta = 0$ or $180°$, then $\sin\theta = 0$

so eigenvalues are $\cos\theta \pm 0$,

i.e., just $\cos\theta$ (which is real).

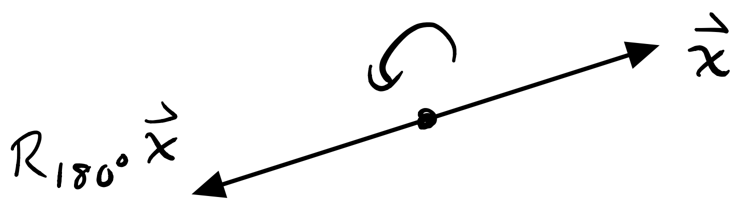
$\theta = 0$: $R_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ = do nothing.

Eigenvalues: 1

Every nonzero vector is a 1-e.vector.

$$E_1 = \mathbb{R}^2.$$

$\theta = 180°$: $R_{180°} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

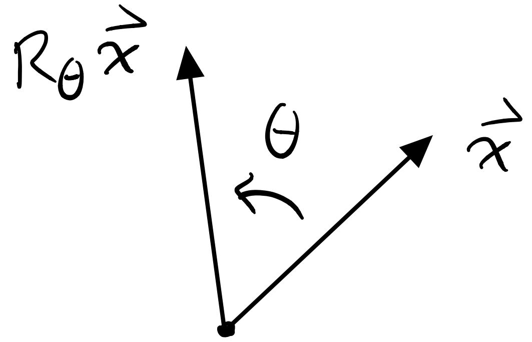


Eigenvalues: -1

Every nonzero vector is (-1)-e.vector

$$E_{-1} = \mathbb{R}^2.$$

If $\theta \neq 0$ & $\theta \neq 180°$ then there are NO (real) EIGENVECTORS, because every vector "changes direction"



There are "complex eigenvectors," but it is not so clear what they mean...

Problem 4: Diagonalization.

$$A = \frac{1}{6}\begin{pmatrix} 5 & 4 \\ 2 & -2 \end{pmatrix}.$$

TRICK: $6A$ has the same e.vectors as A, but with e.values multiplied by 6. Find e.values of $6A$:

$$\det\begin{pmatrix} 5-\lambda & 4 \\ 2 & -2-\lambda \end{pmatrix} = 0$$

$$(5-\lambda)(-2-\lambda) - 8 = 0$$

$$\lambda^2 - 3\lambda - 10 - 8 = 0$$

$$\lambda^2 - 3\lambda - 18 = 0$$

$$\lambda = (3 \pm \sqrt{9+72})/2$$

$$= (3 \pm 9)/2$$

$$= 6 \text{ or } -3.$$

So eigenvalues of A are 1 & $-\frac{1}{2}$.

Eigenvectors of A & $6A$ are the same vectors:

$N(6A - 6I)$

$$= N\begin{pmatrix} 5-6 & 4 \\ 2 & -2-6 \end{pmatrix}$$

$$= N\begin{pmatrix} -1 & 4 \\ 2 & -8 \end{pmatrix}$$

$$= \left\{ \vec{u} : \begin{pmatrix} -1 & 4 \\ 2 & -8 \end{pmatrix}\vec{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$= \text{the line } t\begin{pmatrix} 4 \\ 1 \end{pmatrix}.$

$N(6A + 3I)$

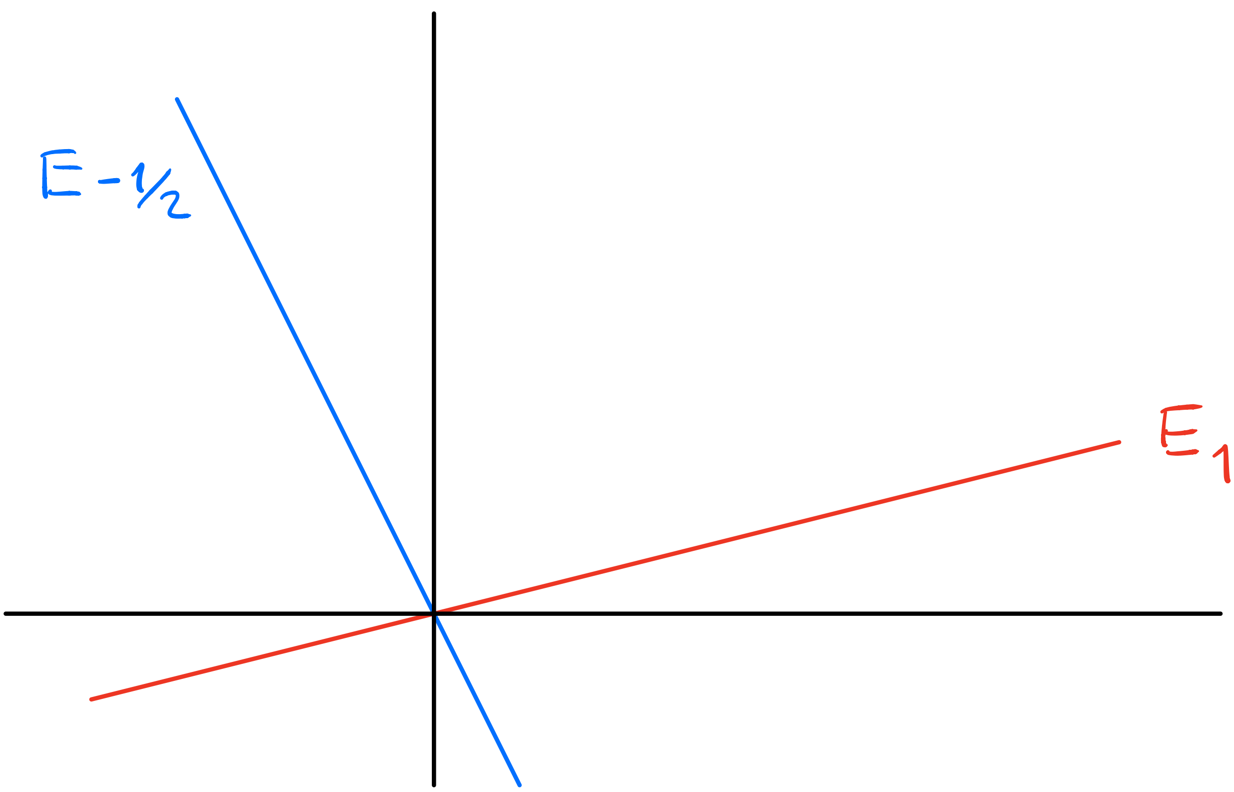
$= N\begin{pmatrix} 5+3 & 4 \\ 2 & -2+3 \end{pmatrix}$

$= N\begin{pmatrix} 8 & 4 \\ 2 & 1 \end{pmatrix}$

$= \left\{ \vec{v} : \begin{pmatrix} 8 & 4 \\ 2 & 1 \end{pmatrix} \vec{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$= \text{the line } t\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$

Picture: Eigenspaces of A



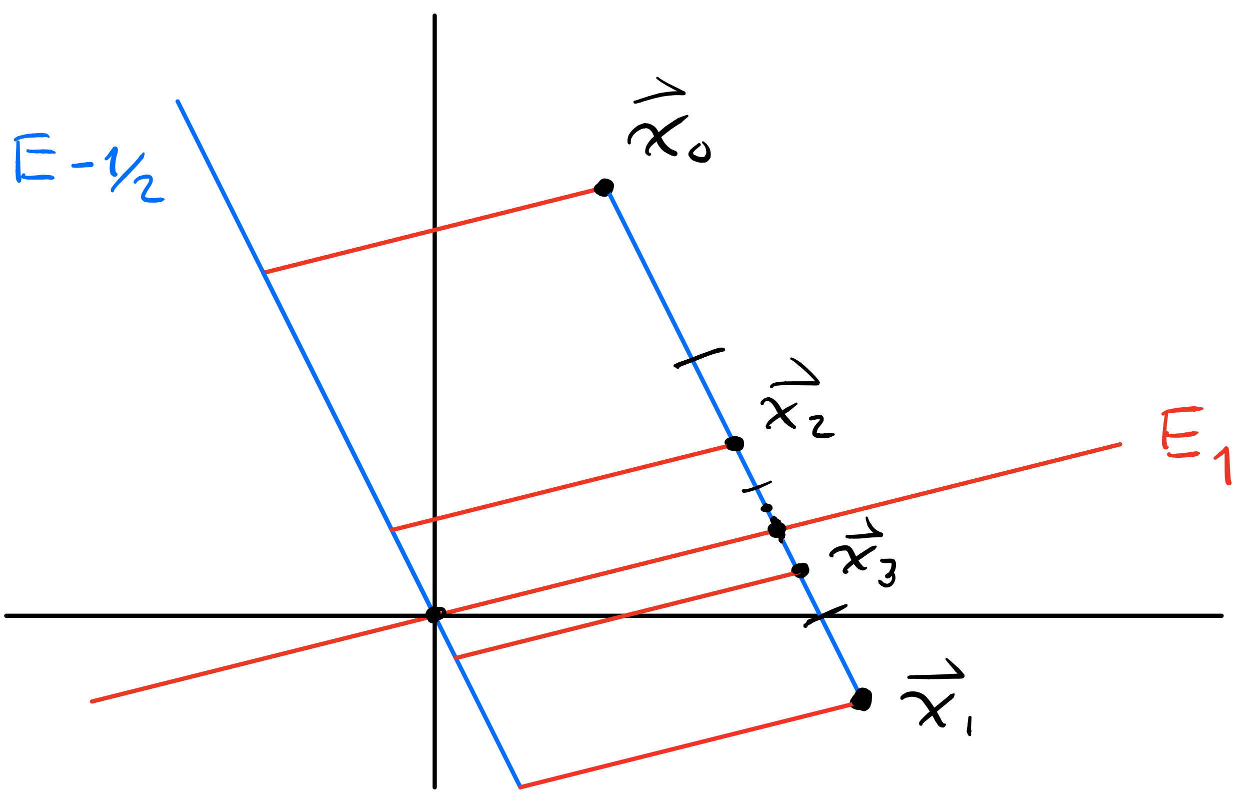
Problem 5: Applications.

Solve the recurrence

$$\vec{x}_0 = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad \& \quad \vec{x}_{n+1} = A\vec{x}_n$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 5 & 4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Solve for x_n & y_n. Picture:



At each step: Red coordinate gets multiplied by 1, blue coordinate gets multiplied by $-1/2$. We <u>see</u> that

$$\vec{x}_n \to \begin{pmatrix} 4 \\ 1 \end{pmatrix} \text{ as } n \to \infty.$$

Computations:

Express $\vec{x}_0 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ in terms of eigenvectors.

$$a\begin{pmatrix} 4 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{-9}\begin{pmatrix} -2 & -1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 2 \\ 5 \end{pmatrix} = 1\begin{pmatrix} 4 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

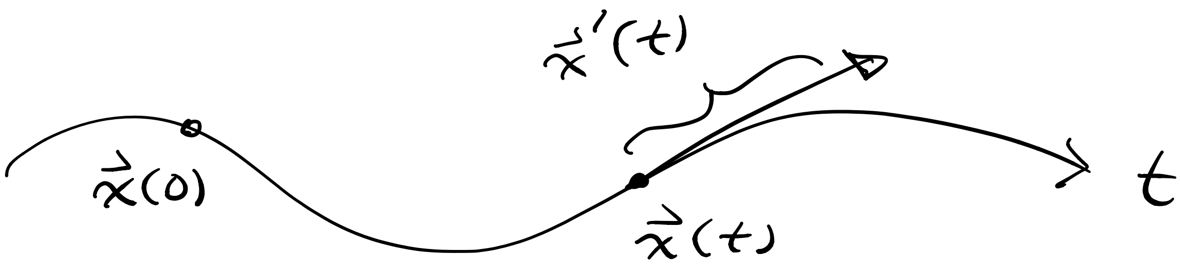
$$\begin{pmatrix}2\\5\end{pmatrix} = \underbrace{\begin{pmatrix}4\\1\end{pmatrix}}_{1\text{-eigenvector}} + \underbrace{\begin{pmatrix}-2\\4\end{pmatrix}}_{-\frac{1}{2}\text{-eigenvector}}$$

Solution:

$$\begin{aligned}
\vec{x}_n &= A^n \vec{x}_0 \\
&= A^n\left(\begin{pmatrix}4\\1\end{pmatrix} + \begin{pmatrix}-2\\4\end{pmatrix}\right) \\
&= A^n\begin{pmatrix}4\\1\end{pmatrix} + A^n\begin{pmatrix}-2\\4\end{pmatrix} \\
&= 1^n\begin{pmatrix}4\\1\end{pmatrix} + \left(-\tfrac{1}{2}\right)^n\begin{pmatrix}-2\\4\end{pmatrix} \\
&= \begin{pmatrix}4 - 2(-1/2)^n \\ 1 + 4(-1/2)^n\end{pmatrix} \rightsquigarrow \begin{pmatrix}4\\1\end{pmatrix}
\end{aligned}$$

as $n \rightsquigarrow \infty$.

A continuous dynamical system:



Suppose the velocity of the path is defined by the matrix A:

$$\vec{x}'(t) = A\vec{x}(t)$$

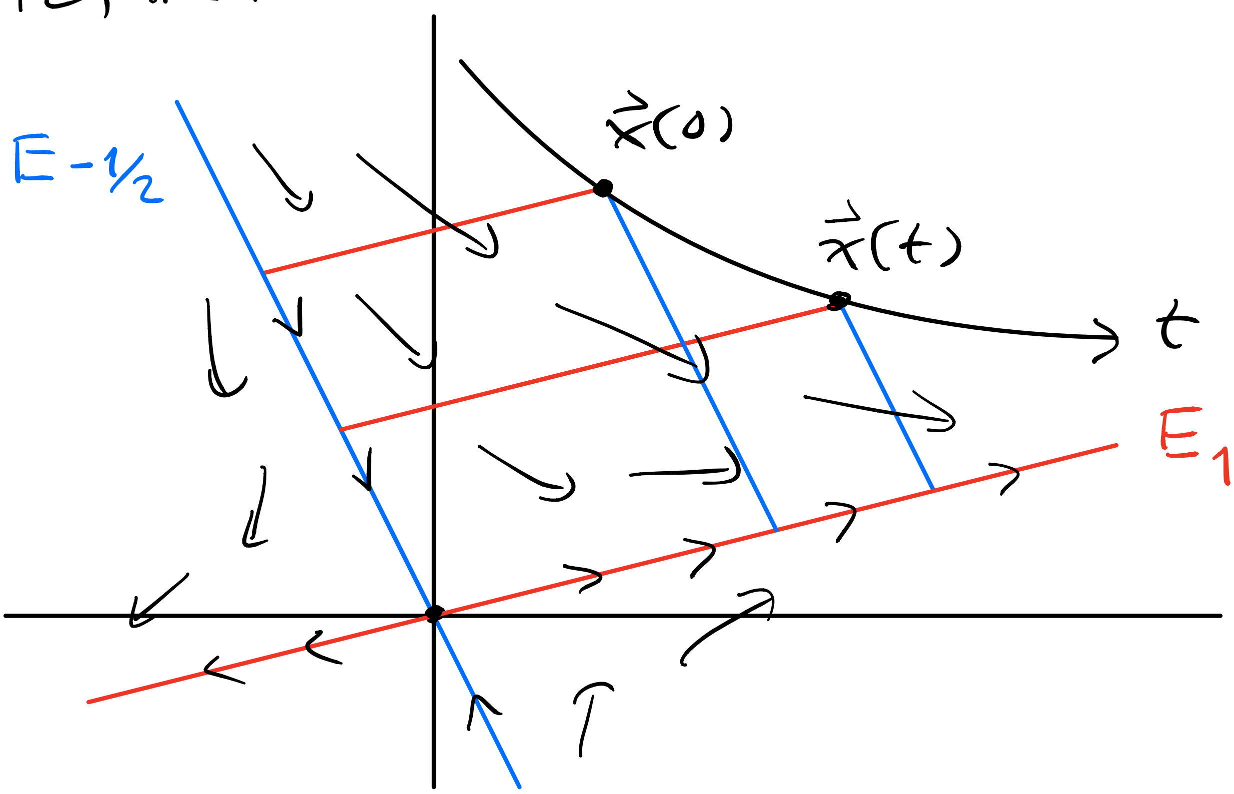
$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 5 & 4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\begin{cases} x'(t) = (5x(t) + 4y(t))/6, \\ y'(t) = (2x(t) - 2y(t))/6. \end{cases}$$

Given initial position $\vec{x}(0) = \binom{2}{5}$,

Find the functions $x(t)$ & $y(t)$.

Picture:



So we know what it looks like.

But how do we compute it?

The exponential function is defined by a "power series":

$$e^x := 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots + \frac{1}{n!}x^n + \cdots$$

Differentiate term by term:

$$\frac{d}{dx}e^x = 0 + 1 + x + \frac{1}{2}x^2 + \cdots$$

$$= e^x \quad \text{(miracle!)}$$

Let's also compute

$$\frac{d}{dt}e^{at} = \frac{d}{dt}\left(1 + ta + \frac{t^2}{2}a^2 + \frac{t^3}{6}a^3 + \cdots\right)$$

$$= 0 + a + ta^2 + \frac{t^2}{2}a^3 + \cdots$$

$$= a\left(1 + ta + \frac{t^2}{2}a^2 + \cdots\right)$$

$= a e^{at}$ (miracle!)

Consequence: Differential equation

$$x'(t) = a\, x(t)$$

has the solution

$$x(t) = c\, e^{at}$$

for some constant c. To find c we plug in $t = 0$:

$$x(0) = c\, e^{0} = c.$$

$$\Longrightarrow x(t) = x(0)\, e^{at}.$$

We can do exactly the same thing for matrices. Given square matrix A we define the "exponential matrix":

$$e^{A} = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots$$

Now d/dA makes no sense,

but $d/dt\, e^{At}$ <u>still</u> makes sense:

$$e^{At} = I + tA + \frac{t}{2}A^2 + \frac{t^3}{6}A^3 + \cdots$$

$$\frac{d}{dt}e^{At} = 0 + A + tA^2 + \frac{t}{2}A^3 + \cdots$$

$$= A\left(I + tA + \frac{t}{2}A^2 + \cdots\right)$$

$$= A\,e^{At} \quad (\text{miracle!})$$

Consequence: The system of differential equations

$$\vec{x}\,'(t) = A\,\vec{x}(t)$$

has the solution

$$\vec{x}(t) = e^{At}\,\vec{x}(0).$$

Problem: It might be very hard to find the matrix e^{At}. But luckily we don't need to!

Recall the important formula from Problem 1.

If $A\vec{u} = \lambda\vec{u}$ then $f(A)\vec{u} = f(\lambda)\vec{u}$ for any polynomial $f(x)$. But a power series is just like a polynomial, so we get

$$\boxed{e^{At}\vec{u} = e^{\lambda t}\vec{u}}$$

In other words: If $\vec{u}$ is a λ-eigenvector of matrix A, then $\vec{u}$ is a $e^{\lambda t}$-eigenvector of the exponential matrix e^{At}.

Continuous Dynamical Systems:

Given square matrix A, suppose we can find

$$A\vec{u} = \lambda\vec{u}$$
$$A\vec{v} = \mu\vec{v}$$
$$\vec{x}(0) = a\vec{u} + b\vec{v}.$$

Then the differential equations

$$\vec{x}'(t) = A\vec{x}(t)$$

have the solution

$$\begin{aligned}\vec{x}(t) &= e^{At}\vec{x}(0)\\ &= e^{At}(a\vec{u}+b\vec{v})\\ &= a\,e^{At}\vec{u} + b\,e^{At}\vec{v}\\ &= a\,e^{\lambda t}\vec{u} + b\,e^{\mu t}\vec{v}.\end{aligned}$$

In our case:

$$\begin{aligned}\vec{x}(t) &= e^{At}\begin{pmatrix}2\\5\end{pmatrix}\\ &= e^{At}\left(\begin{pmatrix}4\\1\end{pmatrix}+\begin{pmatrix}-2\\4\end{pmatrix}\right)\\ &= e^{At}\begin{pmatrix}4\\1\end{pmatrix} + e^{At}\begin{pmatrix}-2\\4\end{pmatrix}\end{aligned}$$

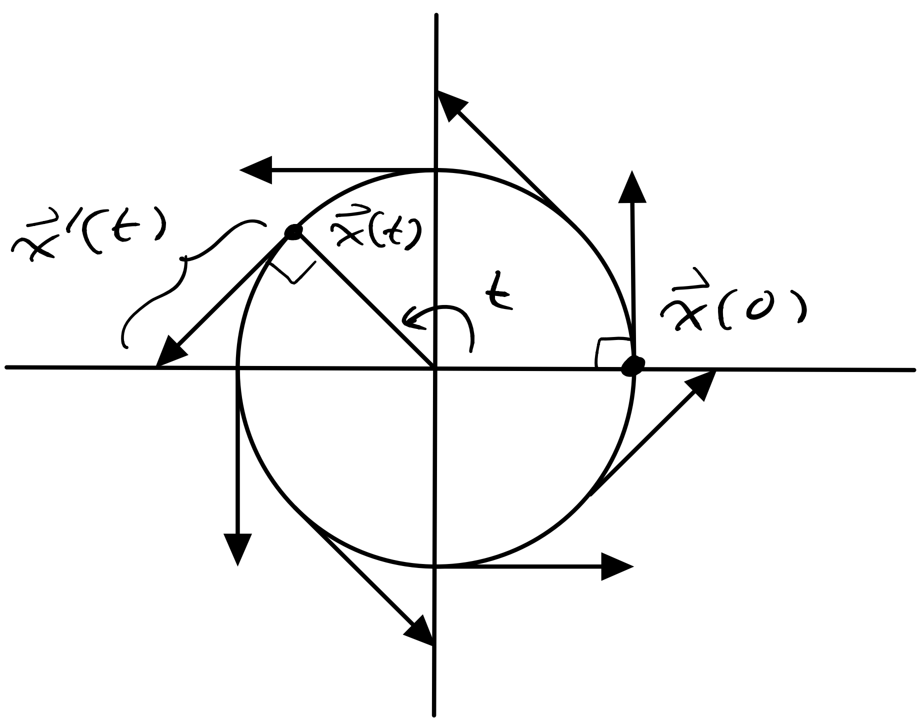
$$= e^{1t}\begin{pmatrix}4\\1\end{pmatrix} + e^{-t/2}\begin{pmatrix}-2\\4\end{pmatrix}$$

$$= \begin{pmatrix}4e^{t} - 2e^{-t/2}\\ e^{t} + 4e^{-t/2}\end{pmatrix}.$$

That's pretty good!

Grand Finale: Consider the rotation matrix $A = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$.

This defines the following vector field:



If we start at $\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

where do we go?

Guess: $\vec{x}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$.

Check:

$$\vec{x}'(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

So this is correct, but what does the eigenvector method say?

Diagonalize $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Eigenvalues:

$$\det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda^2 = -1$$

$$\lambda = \pm i.$$

Eigenvectors:

$$E_i = N\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}$$

$$= \left\{ \vec{u} : \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \vec{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$= \text{the "line" } t\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$E_{-i} = N\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}$$

$$= \left\{ \vec{v} : \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \vec{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$= \text{the "line" } t\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

These are some strange kind of "imaginary lines." But they are still useful.

Express $\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in terms of eigenvectors:

$$\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\-i\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\i\end{pmatrix}$$

So the solution of the dynamical system is:

$$\vec{x}(t) = e^{At}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= e^{At}\left(\frac{1}{2}\begin{pmatrix}1\\-i\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\i\end{pmatrix}\right)$$

$$= \frac{1}{2}e^{At}\begin{pmatrix}1\\-i\end{pmatrix} + \frac{1}{2}e^{At}\begin{pmatrix}1\\i\end{pmatrix}$$

$$= \frac{1}{2}e^{it}\begin{pmatrix}1\\-i\end{pmatrix} + \frac{1}{2}e^{-it}\begin{pmatrix}1\\i\end{pmatrix}$$

$$= \begin{pmatrix}(e^{it} + e^{it})/2\\(-e^{it} + e^{-it})i/2\end{pmatrix}.$$

So that's the answer. But does this equal $\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}$?

Yes it does!

Euler's Formula:

$$e^{it} = \cos t + i \sin t$$
$$e^{-it} = \cos t - i \sin t$$

Add the equations:

$$e^{it} + e^{-it} = 2 \cos t$$
$$\cos t = (e^{it} + e^{-it})/2$$

Subtract the equations:

$$e^{it} - e^{-it} = 2i \sin t$$
$$\sin t = (e^{it} - e^{-it})/2i$$
$$= (-e^{-it} + e^{-it})i/2$$

because $1/i = -i$.

So everything works out. :)

More generally, we expect that

$$\vec{x}(t) = e^{At}\,\vec{x}(0)$$

should rotate the initial position by angle t. In other words, we should have

$$e^{At} = \text{rotation by } t$$

$$e^{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

$$e^{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t} = \cos t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Note that this becomes Euler's Formula after we replace

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ by } 1$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ by } i.$$

So it must be true.

THE END.