HW 1 due next Thurs Sept 3, <u>before</u> the lecture.

Questions?

2(a) What is the midpoint of two points $\vec{u}, \vec{v} \in \mathbb{R}^2$?

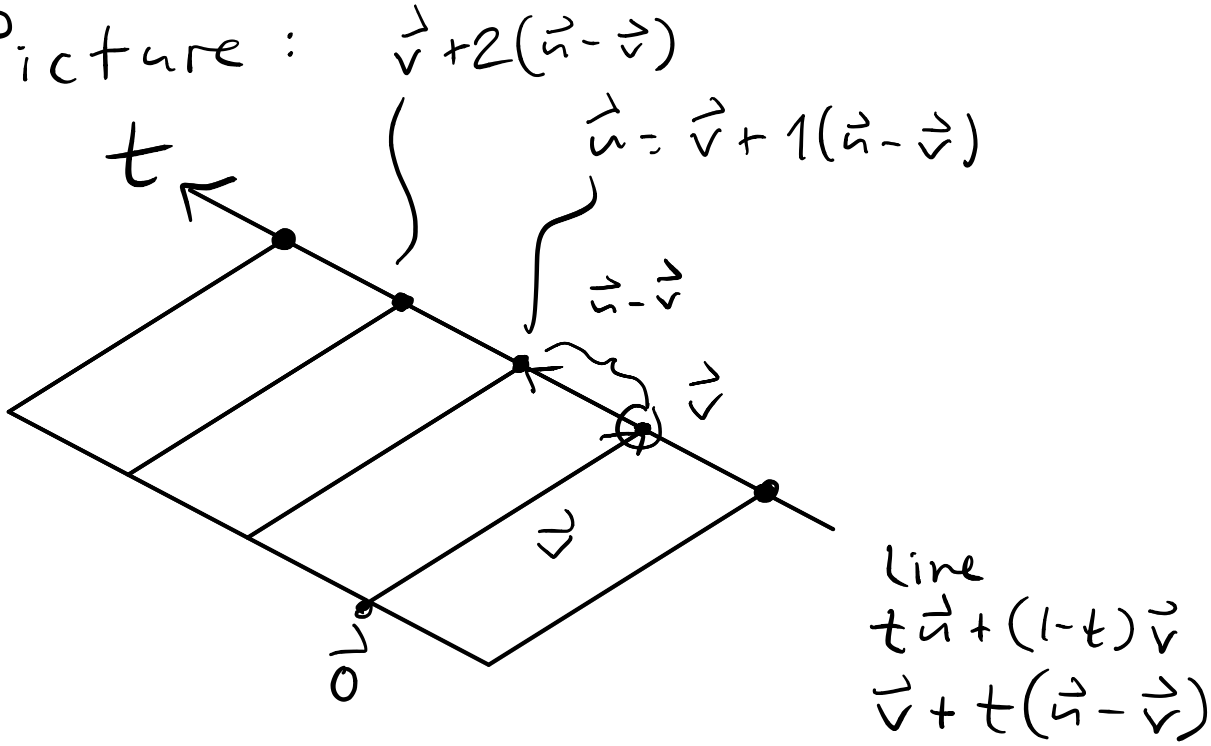
Idea: $\dfrac{\vec{u}+\vec{v}}{2}$?

Technically this is

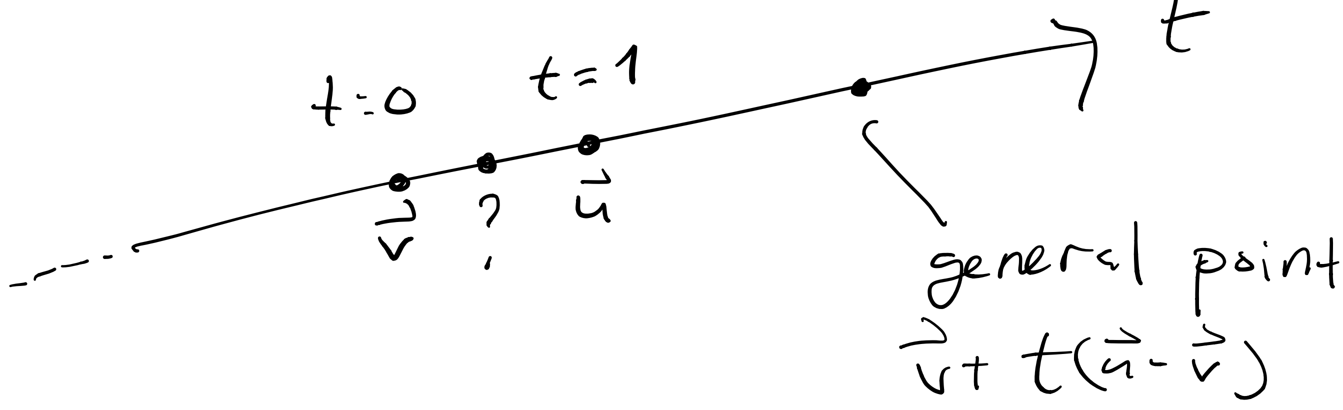
$$\frac{1}{2}\vec{u} + \frac{1}{2}\vec{v}$$

(scalar)(vector) + (scalar)(vector) = vector.

Picture:



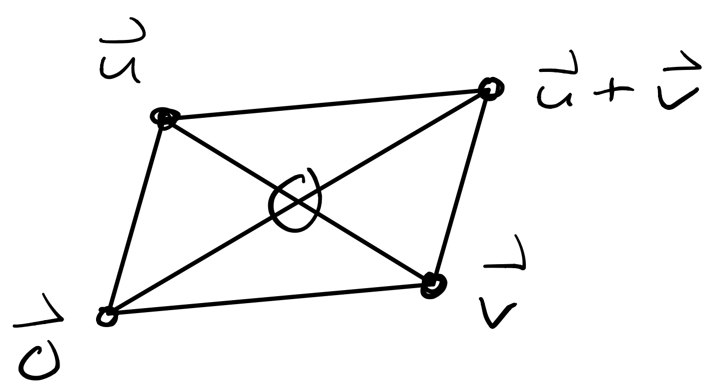
Cleaner:



So the midpoint should have t-coordinate $t = 1/2$.

$$\text{midpoint} = \vec{v} + \frac{1}{2}(\vec{u} - \vec{v}) = \frac{1}{2}\vec{u} + \frac{1}{2}\vec{v} \checkmark$$

Another point of view:



Midpoint is the intersection of the two diagonals:

$$t\vec{u} + (1-t)\vec{v} \quad \& \quad s(\vec{u} + \vec{v})$$

For which values of s & t do we have

$t\vec{u} + (1-t)\vec{v} = s(\vec{u} + \vec{v})$?

i.e., "solve" for s & t.

2(b): What is the "midpoint" of 3 points in the plane? How to compute?

Philosophy: Every important idea needs to be presented 5 times, from slightly different points of view.

Handout: Rules of Vector Arithmetic.

3 Operations

vector + vector = vector

scalar · vector = vector

vector • vector = scalar.

11 Rules:

e.g. $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$

$(s+t)\vec{x} = s\vec{x} + t\vec{x}$

$\vec{x}\circ\vec{x} \geq 0$ for all $\vec{x}$

$\vec{x}\circ\vec{x} = 0 \iff \vec{x} = \vec{0}$.

Idea: Length

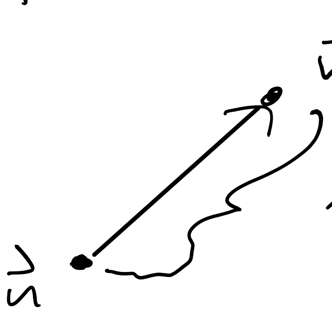
$\|\vec{x}\|^2 = \vec{x}\circ\vec{x}$

- Length is non-negative

- The only vector of length 0 is the zero vector $\vec{0}$

Geometry:

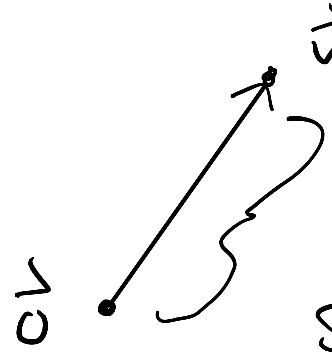
vector is an ordered pair of points



$\vec{x} = \vec{v} - \vec{u}$.

vector = head − tail.

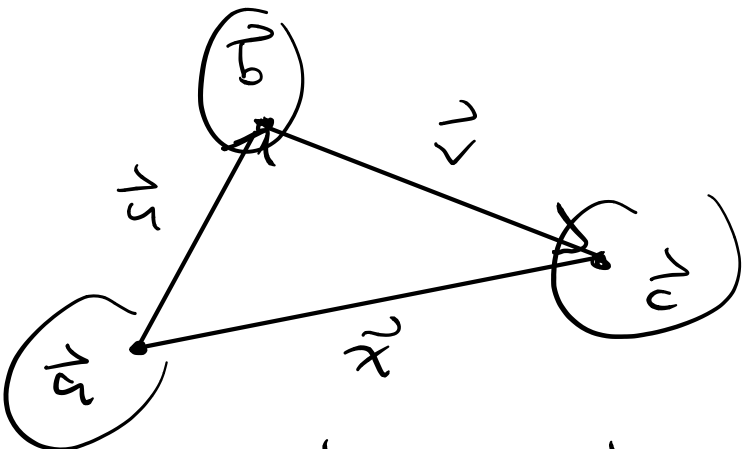
If tail is $\vec{0}$:



vector $= \vec{v} - \vec{0} = \vec{v}$.

Standard Position.

Vectors add "head-to-tail"



Claim: $\vec{x} = \vec{u} + \vec{v}$.

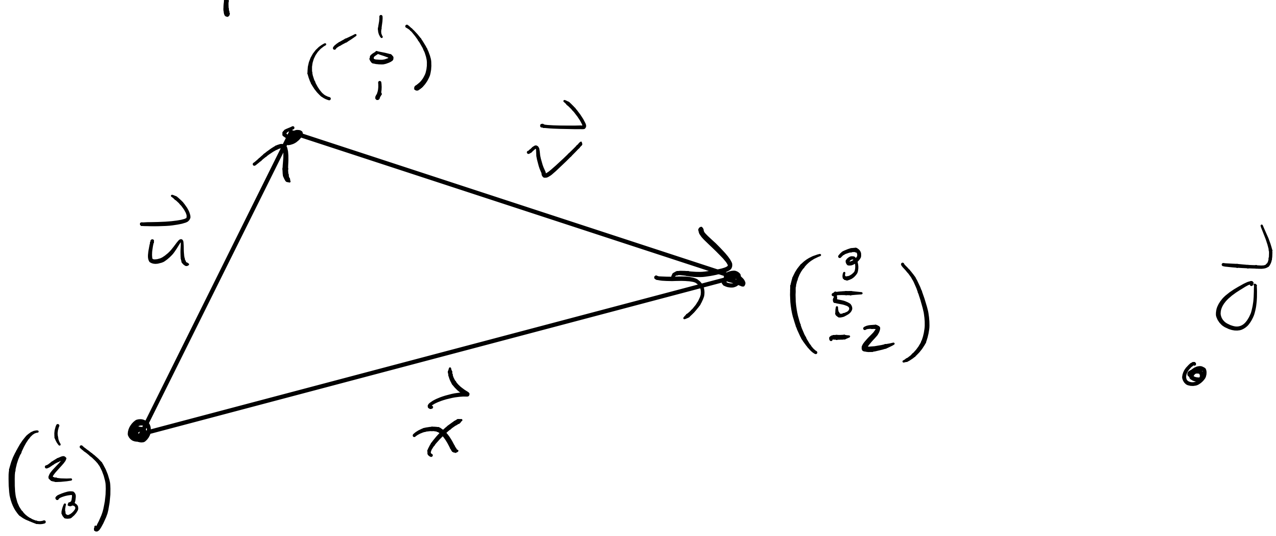
Name the endpoints $\vec{a}, \vec{b}, \vec{c}$

Then $\vec{u} + \vec{v} = (\cancel{\vec{b}} - \vec{a}) + (\vec{c} - \cancel{\vec{b}})$

$= \vec{c} - \vec{a}$

$= \vec{x}$ ✓

Example:



$$\vec{u} = \begin{pmatrix}-1\\0\\1\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}-2\\-2\\-2\end{pmatrix}$$

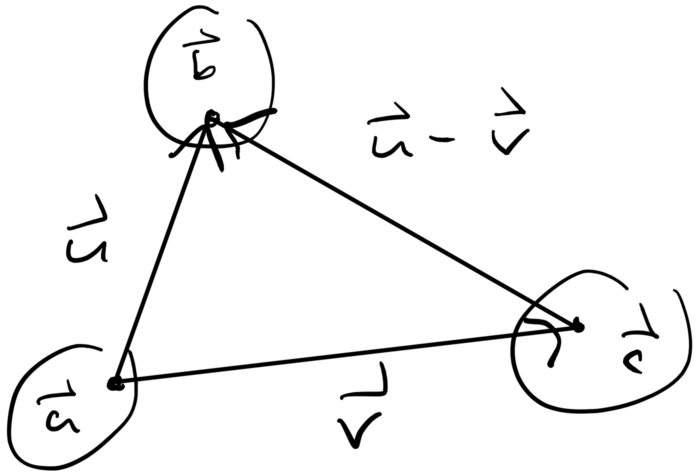
$$\vec{v} = \begin{pmatrix}3\\5\\-2\end{pmatrix} - \begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}4\\5\\-3\end{pmatrix}$$

$$\vec{x} = \begin{pmatrix}3\\5\\-2\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}2\\3\\-5\end{pmatrix}$$

Check:

$$\vec{u} + \vec{v} = \begin{pmatrix}-2\\-2\\-2\end{pmatrix} + \begin{pmatrix}4\\5\\-3\end{pmatrix} = \begin{pmatrix}2\\3\\-5\end{pmatrix} \checkmark$$

Applies to Subtraction:



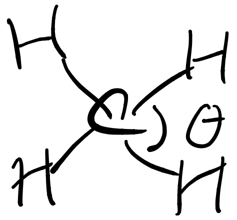
Check: $\vec{u}-\vec{v} = (\vec{b}-\cancel{\vec{a}}) - (\vec{c}-\cancel{\vec{a}})$

$$= \vec{b}-\vec{c} \quad \checkmark$$

We used this triangle last time to compute the angle θ between $\vec{u}$ & $\vec{v}$:

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta .$$

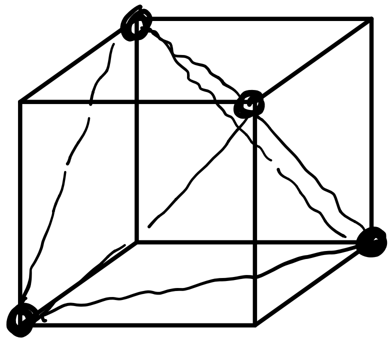
Thinking Homework from last time:



Tetrahedral Angle

$\theta \approx 109.5^\circ$

Why? Find a coordinate system.



Center at $(0,0,0)$

8 vertices at

$(\pm 1, \pm 1, \pm 1)$

Half of the vertices form a regular tetrahedron. 2 choices:

$(1,1,1)$

$(1,-1,-1)$

$(-1,1,-1)$

$(-1,-1,1)$

OR

$(-1,-1,-1)$

$(-1,1,1)$

$(1,-1,1)$

$(1,1,-1)$

Put carbon at $(0,0,0)$

Put hydrogen at these 4 points.

Angle between 2 hydrogen?

All the same; pick 2:

$\vec{u} = (1,1,1)$ & $\vec{v} = (1,-1,-1)$.

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\sqrt{\vec{u}\cdot\vec{u}}\,\sqrt{\vec{v}\cdot\vec{v}}}$$

$$\vec{u}\cdot\vec{v} = 1\cdot 1 + 1(-1) + 1(-1) = -1$$
$$\vec{u}\cdot\vec{u} = 1^2 + 1^2 + 1^2 = 3$$
$$\vec{v}\cdot\vec{v} = 1^2 + (-1)^2 + (-1)^2 = 3$$

$$\cos\theta = \frac{-1}{\sqrt{3}\,\sqrt{3}} = -\frac{1}{3}$$

$$\theta = \arccos\left(-\frac{1}{3}\right)$$
$$\approx 109.471221^\circ$$

[On Quiz you can always leave arccos unevaluated.]

2D Example:

$\vec{u} = (3,1)$ & $\vec{v} = (1,2)$

compute the angle.

$$\vec{u} \circ \vec{v} = 3 \cdot 1 + 1 \cdot 2 = 5$$

$$\vec{u} \circ \vec{u} = 3^2 + 1^2 = 10$$

$$\vec{v} \circ \vec{v} = 1^2 + 2^2 = 5$$

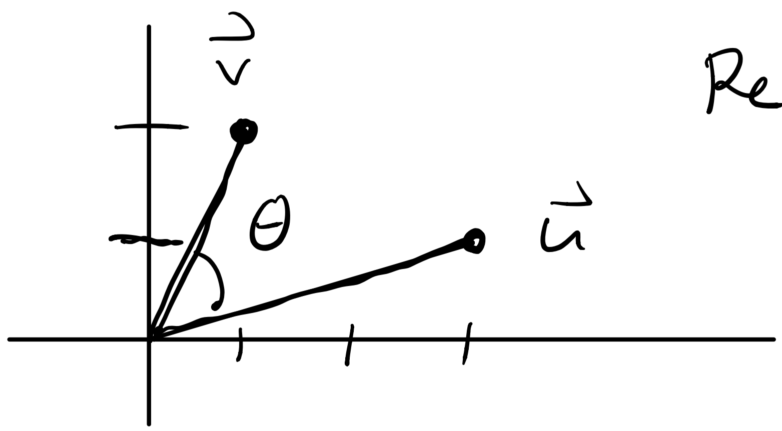
$$\cos\theta = \frac{\vec{u} \circ \vec{v}}{\sqrt{\vec{u} \circ \vec{u}}\ \sqrt{\vec{v} \circ \vec{v}}} = \frac{5}{\sqrt{10}\ \sqrt{5}}$$

$$= \frac{\sqrt{5}}{\sqrt{10}} = \sqrt{5/10} = \sqrt{1/2}$$

$$\left[\text{Remark: } \frac{a}{\sqrt{a}} = \sqrt{a} \right]$$

$$\Longrightarrow \theta = 45^\circ.$$

Picture:



Reasonable ✓

Now let $\vec{x}, \vec{y} \in \mathbb{R}^{100}$ such that

$$\vec{x}\cdot\vec{x} = 1$$
$$\vec{y}\cdot\vec{y} = 1$$
$$\vec{x}\cdot\vec{y} = 0.$$

Compute the angle between

$$\vec{u} = 3\vec{x} + \vec{y} \quad \& \quad \vec{v} = \vec{x} + 2\vec{y}.$$

Same Method:

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\sqrt{\vec{u}\cdot\vec{u}}\ \sqrt{\vec{v}\cdot\vec{v}}}.$$

Compute:

$$\vec{u}\cdot\vec{v} = (3\vec{x} + \vec{y})\cdot(\vec{x} + 2\vec{y})$$
$$= 3\,\vec{x}\cdot\vec{x} + 6\,\vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{x} + 2\,\vec{y}\cdot\vec{y}$$
$$= 3\,\underbrace{\cancel{\vec{x}\cdot\vec{x}}}_{1} + 7\,\underbrace{\cancel{\vec{x}\cdot\vec{y}}}_{0} + 2\,\underbrace{\cancel{\vec{y}\cdot\vec{y}}}_{1}$$
$$= 3 + 0 + 2 = 5.$$

$$\vec{u} \cdot \vec{u} = (3\vec{x} + \vec{y}) \cdot (3\vec{x} + \vec{y})$$

$$= 9 \underbrace{\cancel{\vec{x} \cdot \vec{x}}}_{1} + 6 \underbrace{\cancel{\vec{x} \cdot \vec{y}}}_{0} + 1 \underbrace{\cancel{\vec{y} \cdot \vec{y}}}_{1}$$

$$= 9 + 0 + 1 = 10.$$

$$\vec{v} \cdot \vec{v} = (\vec{x} + 2\vec{y}) \cdot (\vec{x} + 2\vec{y})$$

$$= \underbrace{\cancel{\vec{x} \cdot \vec{x}}}_{1} + 4 \underbrace{\cancel{\vec{x} \cdot \vec{y}}}_{0} + 4 \underbrace{\cancel{\vec{y} \cdot \vec{y}}}_{1}$$

$$= 1 + 0 + 4 = 5.$$

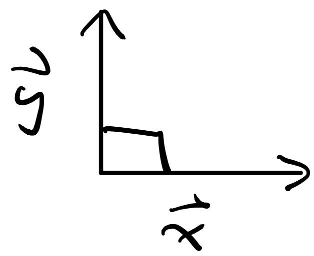
Hence

$$\cos\theta = \frac{5}{\sqrt{10}\,\sqrt{5}} = \sqrt{\frac{1}{2}}$$

$$\theta = 45^\circ \quad \text{AGAIN!}$$

Why is it the same?

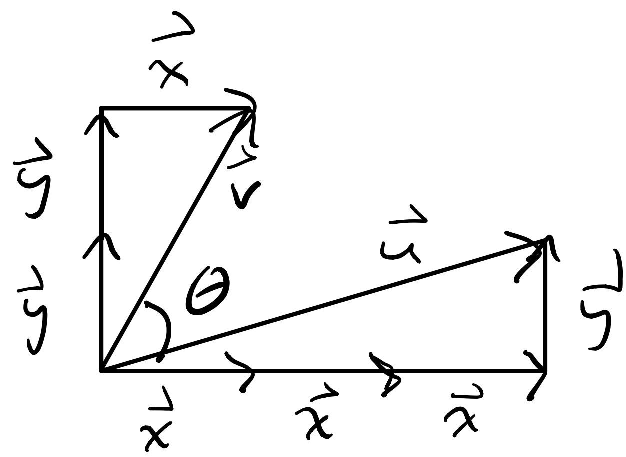
Idea: $\vec{x}$ & $\vec{y}$ form a little 2D coordinate system living in 100D space.



$$\vec{x}\circ\vec{x}=1 \Rightarrow \|\vec{x}\|=1$$
$$\vec{y}\circ\vec{y}=1 \Rightarrow \|\vec{y}\|=1$$
$$\vec{x}\circ\vec{y}=0 \Rightarrow \vec{x}\perp\vec{y}.$$

[Remark : Given $\vec{a}$ & $\vec{b}$ we have
$\vec{a}\perp\vec{b}$ (perpendicular)
$\iff \cos\theta = 0$
$\iff \vec{a}\circ\vec{b}=0.$]

We can draw a picture of
$\vec{u}=3\vec{x}+\vec{y}$ & $\vec{v}=\vec{x}+2\vec{y}$:



Reasonable ✓

Moral: Working with the coord. system $\vec{x}$ & $\vec{y}$ is just like working in the Cartesian plane!

Jargon: If $\vec{x} \circ \vec{x} = \vec{y} \circ \vec{y} = 1$
and $\vec{x} \circ \vec{y} = 0$

then we say that $\vec{x}$ & $\vec{y}$ form an "orthonormal coordinate system"

"otho" = "perpendicular"

"normal" = "length 1"

More generally, an "n-dimensional orthonomal coordinate system" is a set of vectors

$$\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$$

such that

$$\begin{cases} \vec{e}_i \bullet \vec{e}_i = 1 & \text{for all } i \\ \vec{e}_i \bullet \vec{e}_j = 0 & \text{for all } i, j \text{ with } i \neq j. \end{cases}$$