HW1 is up! Due Thurs Sept 3 <u>before</u> class. No late HW accepted.

Current Topic: Vector Arithmetic.

Jargon: Let $\mathbb{R}^n$ be the set of column vectors with n real entries:

$$\mathbb{R}^n = \left\{ \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : x_1, x_2, \dots, x_n \in \mathbb{R} \right\}$$

= the set of points of n-dimensional space.

There are 3 natural algebraic operations on this set:

- Addition:

For any $\vec{x}, \vec{y} \in \mathbb{R}^n$ we define

$$\vec{x} + \vec{y} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

- Scalar Multiplication :

For any vector $\vec{x} \in \mathbb{R}^n$ and "scalar" $t \in \mathbb{R}$ ("scalar" just a fancy name for "number") we define

$$t\vec{x} = t\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} tx_1 \\ tx_2 \\ \vdots \\ tx_n \end{pmatrix}.$$

- Dot Product :

For any vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ we define

$$\vec{x} \bullet \vec{y} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \bullet \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n .$$

[Remark :

vector + vector = vector

scalar · vector = vector

vector • vector = scalar

In general, there is no reasonable way to define "multiplication" of vectors:

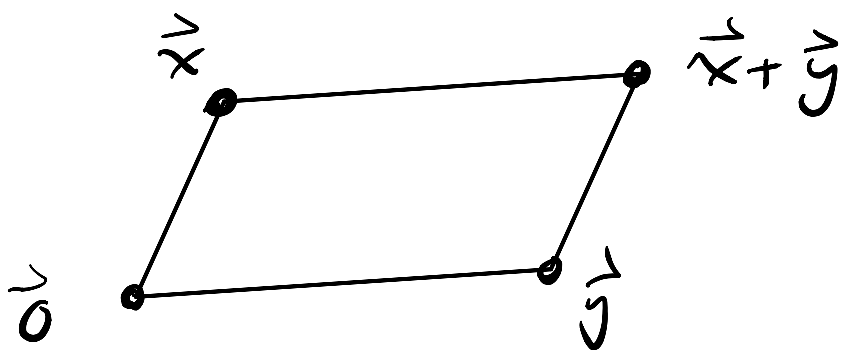
vector × vector = vector ?

NO!

]

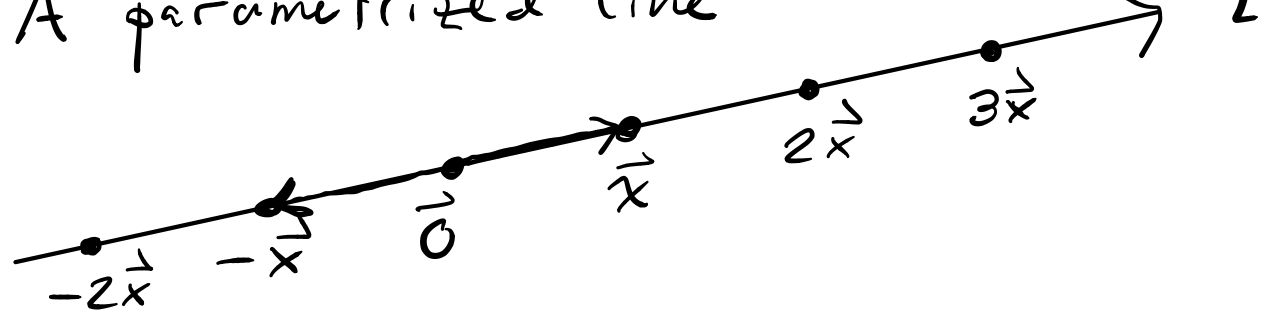
Geometric Meaning:

- Addition: For any $\vec{x}, \vec{y} \in \mathbb{R}^n$ the points $\vec{0}, \vec{x}, \vec{y}, \vec{x}+\vec{y}$ are the vertices of a parallelogram:



- Scalar Mult: Given $\vec{x} \in \mathbb{R}^n$, the vectors $t\vec{x}$ for all $t \in \mathbb{R}$ form the <u>line</u> that is <u>parallel to $\vec{x}$</u> and <u>goes through the origin</u> $\vec{0}$:

A "parametrized line"



Every point on this line corresponds to a unique value of the "parameter" t.

- Dot Product :

Length of vector $\vec{x}$ satisfies

$$\|\vec{x}\|^2 = \vec{x} \cdot \vec{x}$$

$$= x_1 x_1 + x_2 x_2 + \cdots + x_n x_n$$

$$= x_1^2 + x_2^2 + \cdots + x_n^2$$

$$\|\vec{x}\| = +\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

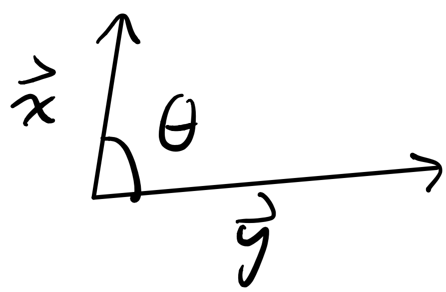
"Pythagorean Theorem"

[Remark : Length vs. Scalar Mult.

I claim that $\|t\vec{x}\| = |t|\,\|\vec{x}\|$.

I'll let you think about this ... }

Angles: Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, let θ be the angle between them, measured tail-to-tail:



Then I claim that

$$\vec{x} \cdot \vec{y} = \|\vec{x}\| \, \|\vec{y}\| \cos\theta .$$

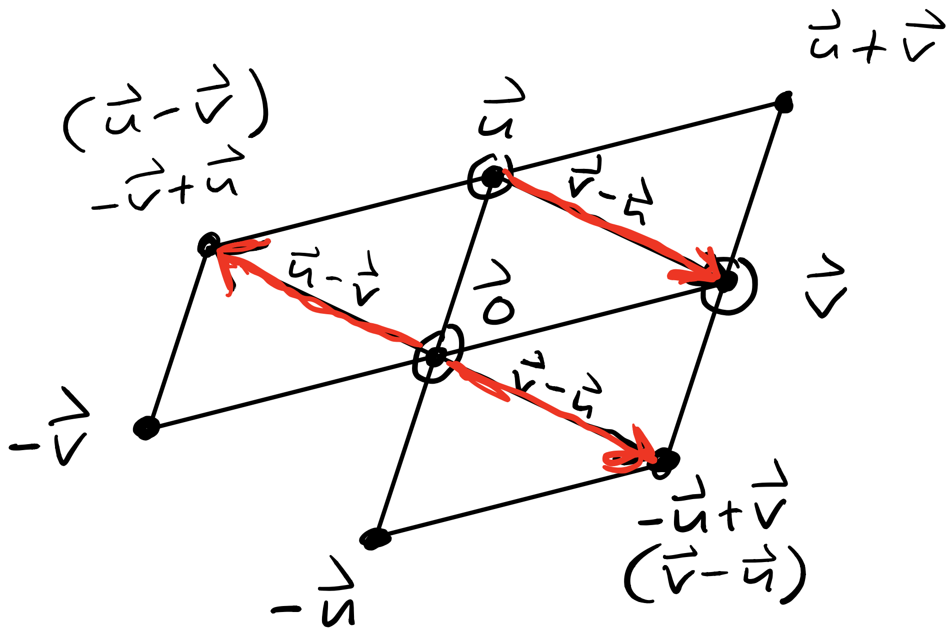
$$\cos\theta = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \, \|\vec{y}\|}$$

$$= \frac{\vec{x} \cdot \vec{y}}{\sqrt{\vec{x} \cdot \vec{x}} \, \sqrt{\vec{y} \cdot \vec{y}}} .$$

Why is this true?

Subtraction of Vectors:

Given vectors $\vec{u}$ & $\vec{v}$, we have the following picture:



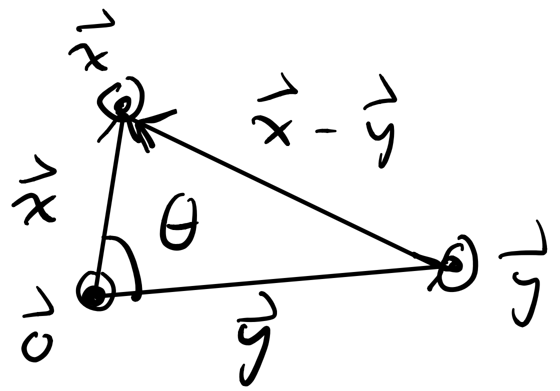
Mnemonic:

Name of a vector = its head − its tail

"vector = head − tail"

Proof of the angle formula:

Given vectors $\vec{x}, \vec{y}$ we have a triangle of vectors:



We want an equation involving θ.
From classical geometry, the Law of Cosines says

$$\| \vec{x} - \vec{y} \|^2 = \| \vec{x} \|^2 + \| \vec{y} \|^2 - 2 \| \vec{x} \| \| \vec{y} \| \cos \theta$$

On the other hand, we can use rules of vector arithmetic to write

$$\| \vec{x} - \vec{y} \|^2 = (\vec{x} - \vec{y}) \bullet (\vec{x} - \vec{y})$$

$$= \vec{x} \cdot \vec{x} - \vec{y} \circ \vec{x} - \vec{x} \cdot \vec{y} + \vec{y} \circ \vec{y}$$

$$= \vec{x} \circ \vec{x} + \vec{y} \circ \vec{y} - 2 \vec{x} \circ \vec{y}$$

Remark: Always $\vec{x} \circ \vec{y} = \vec{y} \circ \vec{x}$.

e.g. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \circ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1 y_1 + x_2 y_2$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = y_1 x_1 + y_2 x_2$

$$= \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2 \vec{x} \circ \vec{y}.$$

In summary, we have two equations:

$$\|\vec{x} - \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\|\vec{x}\|\|\vec{y}\| \cos \theta$$

$$\|\vec{x} - \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2 \vec{x} \circ \vec{y}.$$

Finally, by comparing these equations we obtain

$$\vec{x} \circ \vec{y} = \|\vec{x}\| \|\vec{y}\| \cos \theta \qquad \checkmark$$

Examples:

2D: $\vec{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ & $\vec{y} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

what is the angle?

$$\cos\theta = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|} = \frac{\vec{x} \cdot \vec{y}}{\sqrt{\vec{x} \cdot \vec{x}} \sqrt{\vec{y} \cdot \vec{y}}}.$$

Compute:

$$\vec{x} \cdot \vec{y} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 1 \cdot 3 + 3 \cdot 5$$
$$= 3 + 15 = 18$$

$$\vec{x} \cdot \vec{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 1 \cdot 1 + 3 \cdot 3 = 10.$$

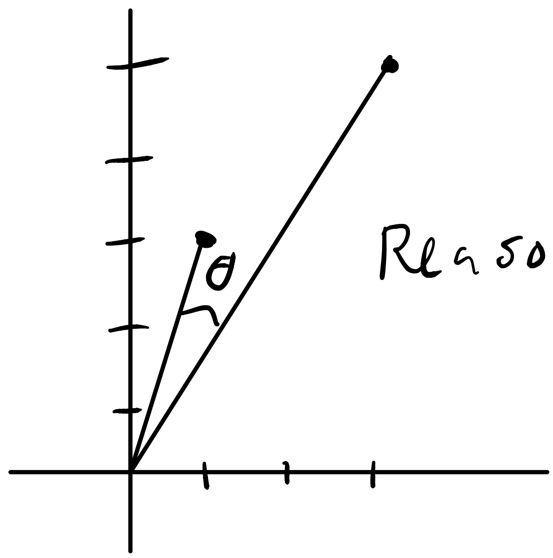
$$\vec{y} \cdot \vec{y} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 3 \cdot 3 + 5 \cdot 5 = 34.$$

$$\cos\theta = \frac{18}{\sqrt{10}\sqrt{34}} \approx 0.976$$

$\theta \approx \arccos(0.976)$

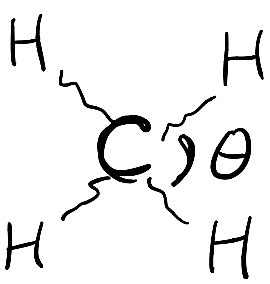
$\approx 12.58°$

Picture :



Reasonable.

3D : "Tetrahedral Angle"



Angle between two hydrogen atoms in methane molecule is

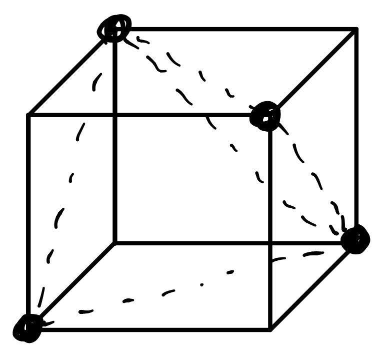
$\theta \approx 109.5°$

But Why ?

Idea: Because of symmetry, H atoms lie at vertices of a "regular tedrahedron" with C atom at the center.

How to put this in coordinates?

Good Trick:



Half the vertices of a cube form a regular tetrahedron.

Thinking Homework:

Finish the calculation...