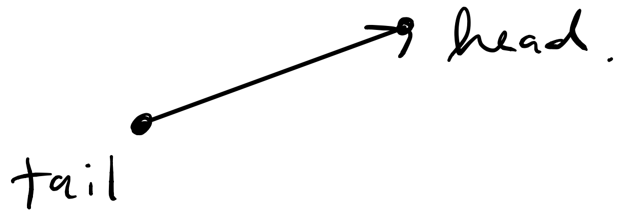
Current Topic: Vector Arithmetic.

What is a vector?

My favorite answer:

An ordered pair of points (head, tail).

Picture:



How can we represent a vector in a coordinate system?

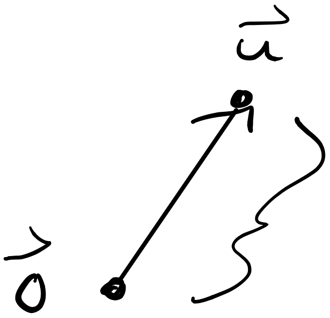
Recall that a point (in n dim space) is a column vector of real numbers:

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}.$$

So, how can we represent an arrow?

Two Rules:

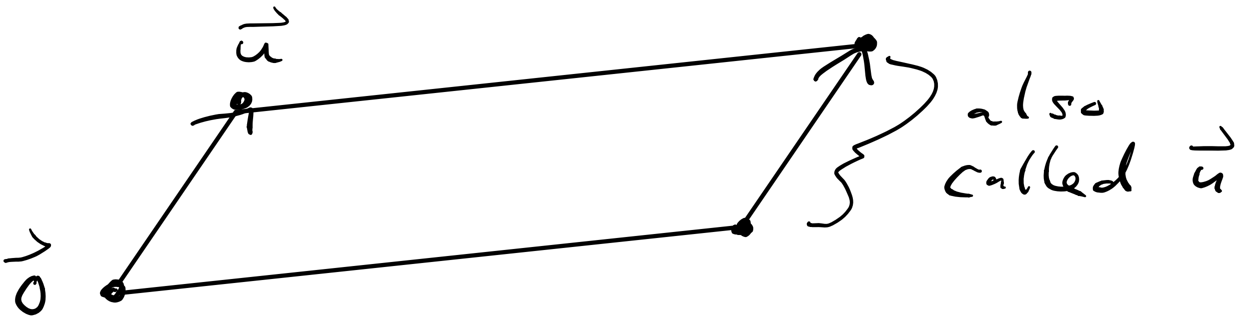
- If the tail is at $\vec{0}$ then name of vector = head.



call this vector $\vec{u}$.

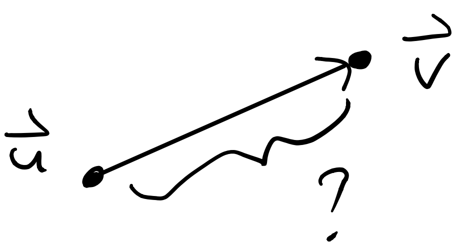
Say this vector is in "standard position."

- If we move a vector, this does not change the name

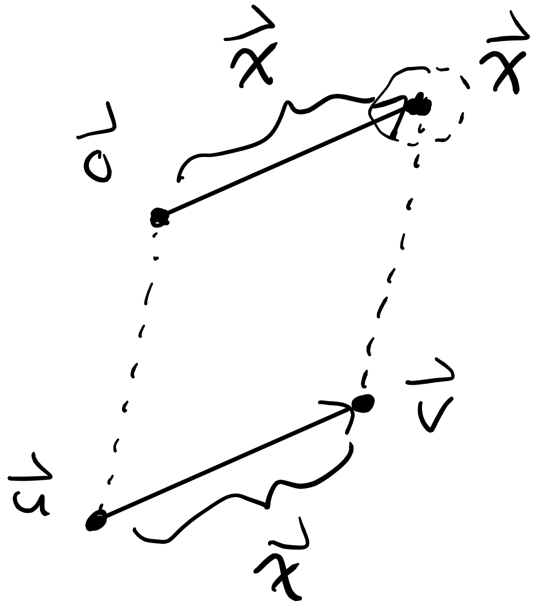


(Everything is built of parallelograms.)

Question: what the name of this vector?



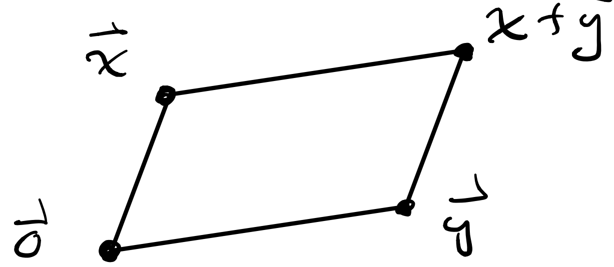
Idea: Put the vector in standard position.



$(\vec{x} + \vec{u} = \vec{v})$

This is a parallelogram, so we know that ...

Recall:



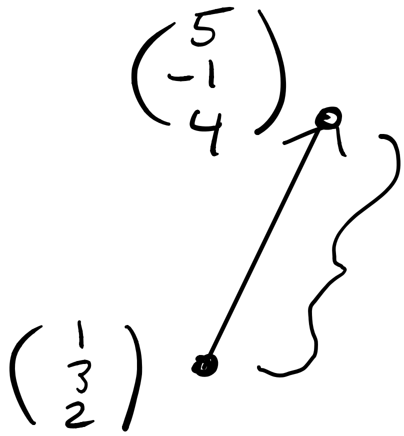
So in our picture we have

$$\vec{x} + \vec{u} = \vec{v}.$$

So $\vec{x} \overset{?}{=} \vec{v} - \vec{u}$?

Can we subtract points? Sure!

Example:



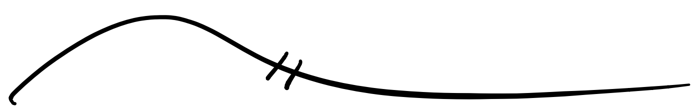
This arrow is called

$$\begin{pmatrix}5\\-1\\4\end{pmatrix} - \begin{pmatrix}1\\3\\2\end{pmatrix} = \begin{pmatrix}4\\-4\\2\end{pmatrix}$$

Mnemonic:

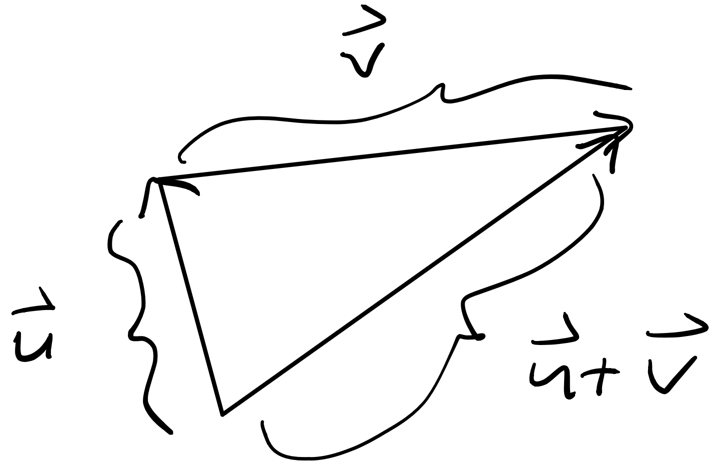
Name of a vector = (its head) − (its tail)

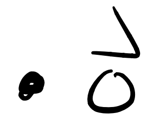
"vector = head − tail"



Why is this good?
Now we don't need to worry about the origin, which is good because the real world doesn't have an origin.

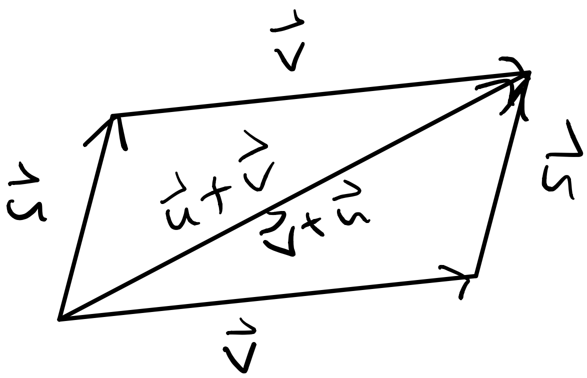
we can express addition in a new way:





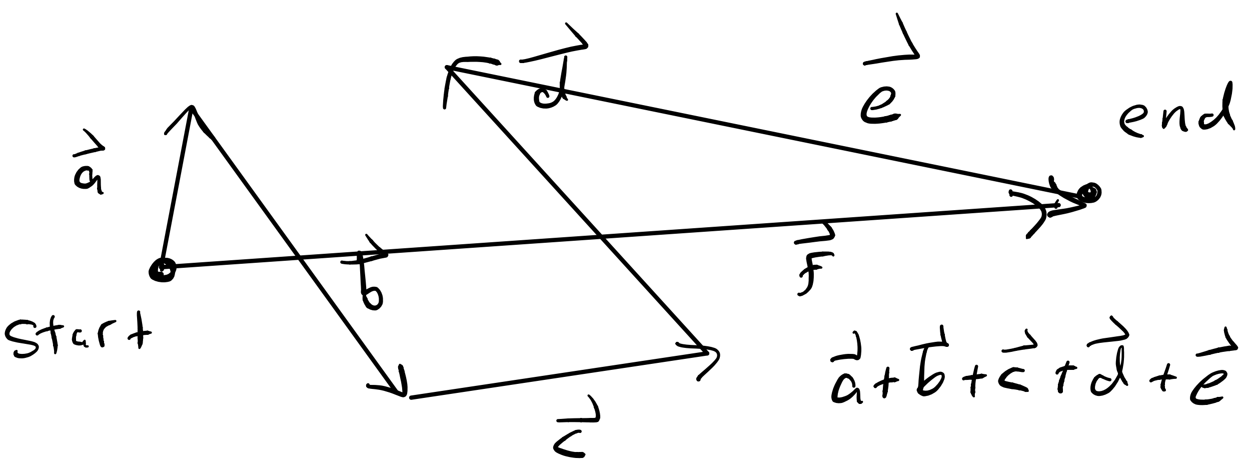
vectors add head-to-tail.

The order doesn't matter:



$\vec{u}+\vec{v}=\vec{v}+\vec{u}$

We can also iterate this procedure:



$\vec{a}+\vec{b}+\vec{c}+\vec{d}+\vec{e}=\vec{f}$

This is why vectors are good for physics.

Newton : Force is a vector.

Now we will discuss

- lengths of vectors
- angles between vectors.

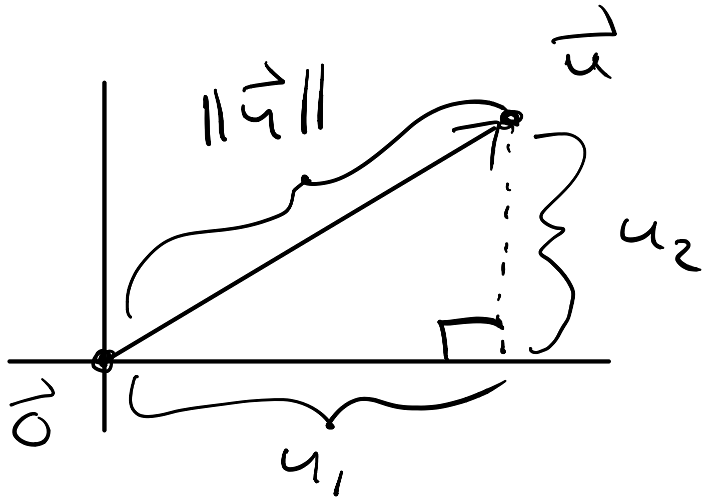
Given $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ in the plane,

let $\|\vec{u}\|$ = the length of the arrow.

= ?

How to compute ?

Put the vector in standard position:



The Pythagorean Theorem says

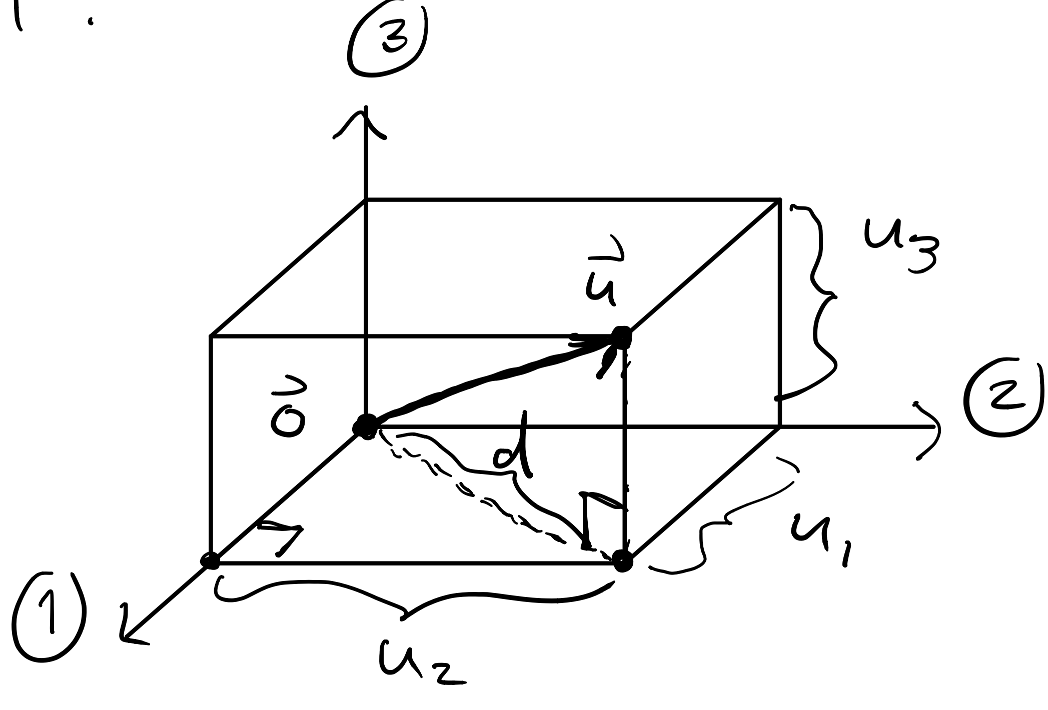
$$\|\vec{u}\|^2 = u_1^2 + u_2^2$$

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2}$$

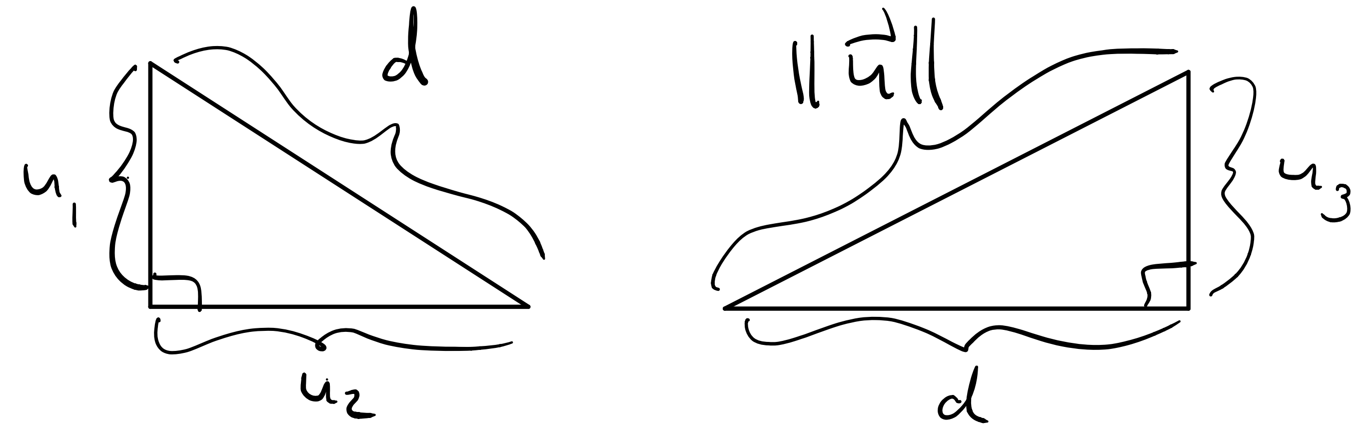
What about 3D? $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$.

Guess: $\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$

Proof:



We have two right triangles:



Pythagoras:

$$d^2 = u_1^2 + u_2^2$$

$$\|\vec{u}\|^2 = d^2 + u_3^2$$

$$= u_1^2 + u_2^2 + u_3^2$$

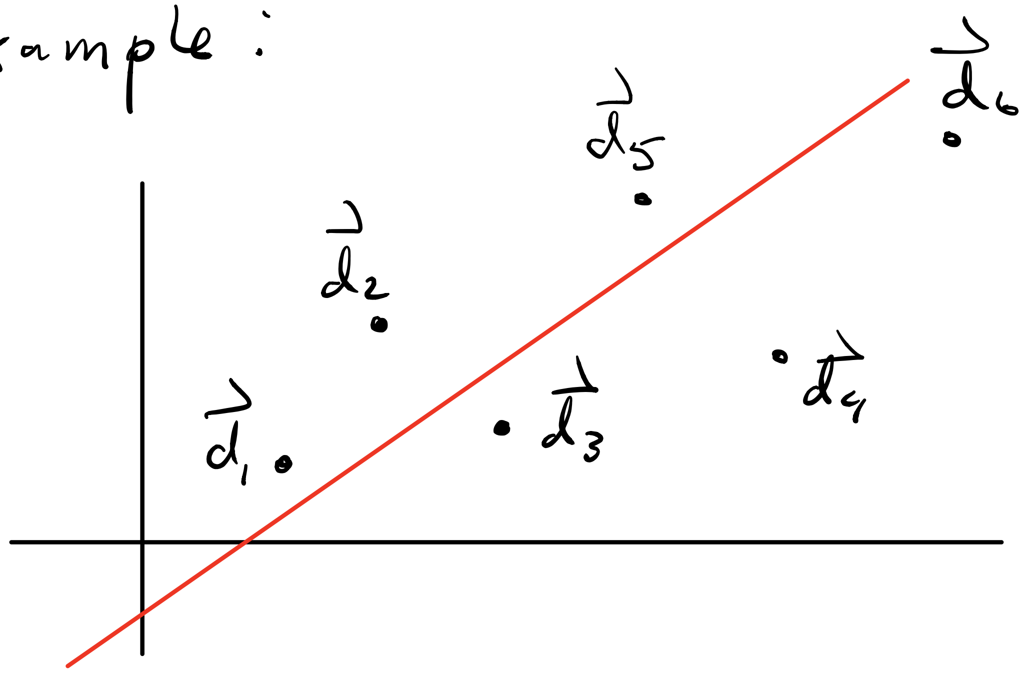
What about 4D, etc?

Let's just say that

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

OK, but why bother? Statistics asks us to compute distances in high dimensional spaces of <u>data</u>.

Example:



Finding the "best fit line" requires us to compute distances in 6D.

(Later...)

Then for any two points $\vec{u}$ & $\vec{v}$, the distance between $\vec{u}$ & $\vec{v}$ is

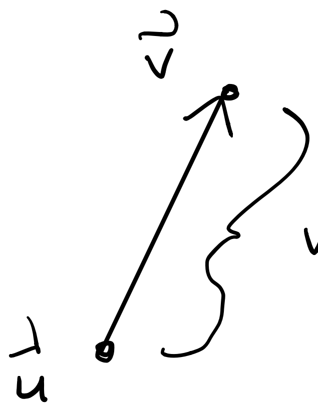
$$\text{dist}(\vec{u},\vec{v}) = \text{length of arrow } \vec{u} \to \vec{v}$$

$$= \| \vec{v} - \vec{u} \|$$

$$= \sqrt{(v_1-u_1)^2 + (v_2-u_2)^2 + \cdots + (v_n-u_n)^2}$$

Easy to compute :)

Example: Distance between $\vec{u} = \begin{pmatrix}1\\2\\3\\4\end{pmatrix}$ & $\vec{v} = \begin{pmatrix}2\\4\\3\\1\end{pmatrix}$.



vector $= \vec{v} - \vec{u}$

$$= \begin{pmatrix}2\\4\\3\\1\end{pmatrix} - \begin{pmatrix}1\\2\\3\\4\end{pmatrix} = \begin{pmatrix}1\\2\\0\\-3\end{pmatrix}$$

$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{v} - \vec{u}\|$$

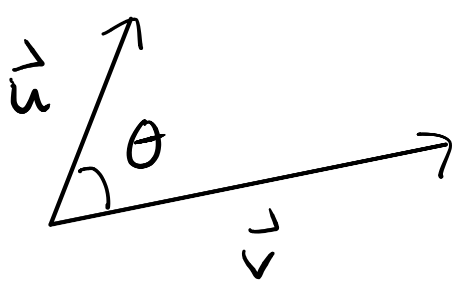
$$= \sqrt{1^2 + 2^2 + 0^2 + (-3)^2}$$

$$= \sqrt{14} \approx 3.74$$

What about angles?

For any two vectors $\vec{u}$ & $\vec{v}$ in n-dimensional space, these vectors live in a 2D plane inside nD.

Picture :



We can measure the angle inside this plane.

Given the coordinates of $\vec{u}$ & $\vec{v}$, how to compute the angle θ ?

The answer is surprising!

We define a new operation called the "dot product" of vectors.

Def : Given $\vec{u}$ & $\vec{v}$ in n-dim space,

let $\vec{u} \bullet \vec{v} := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$.

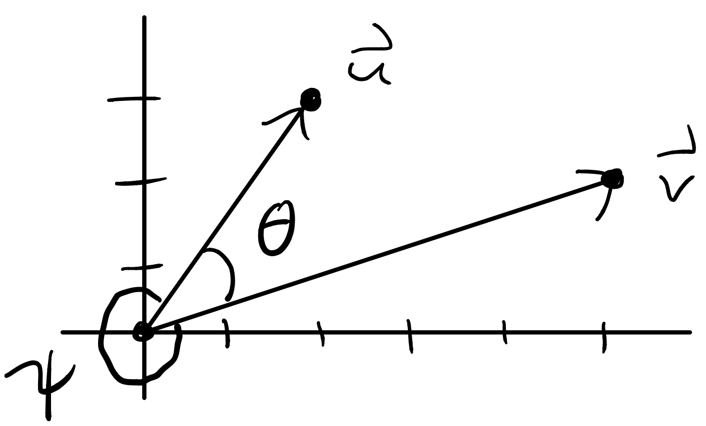
vector $\bullet$ vector $=$ number

Since we're running out of time, I'll tell you the answer and show an example. Next time I'll explain why it works.

Answer: The angle θ between $\vec{u}$ & $\vec{v}$ (measured tail-to-tail) satisfies

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta.$$

Example: Compute the angle between $\vec{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ & $\vec{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$:



First compute

$\|\vec{u}\|$, $\|\vec{v}\|$, $\vec{u} \cdot \vec{v}$.

$$\|\vec{u}\| = \sqrt{2^2+3^2} = \sqrt{13}$$

$$\|\vec{v}\| = \sqrt{5^2+2^2} = \sqrt{29}$$

$$\vec{u}\circ\vec{v} = 2\cdot 5 + 3\cdot 2 = 16.$$

Thus the angle θ satisfies

$$\|\vec{u}\|\|\vec{v}\|\cos\theta = \vec{u}\circ\vec{v}$$

$$\cos\theta = \frac{\vec{u}\circ\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

$$= \frac{16}{\sqrt{13}\sqrt{29}} \approx 0.824$$

Therefore,

$$\theta \approx \arccos(0.824)$$

$$\underset{\substack{\theta \\ \text{small angle}}}{34.5^\circ} \quad \text{or} \quad \underset{\substack{\psi \\ \text{big angle}}}{325.5^\circ} \; (-34.5^\circ)$$