Math 210 Quiz 3

Fall 2020
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You have 20 minutes to write the quiz. No collaboration is allowed. When finished, you have 5 minutes to upload a pdf scan of your work to the google classroom.

Problem 1. [4 points] Consider the following system of linear equations:

$$\left\{\begin{array}{ccccccc} x & + & y & + & z & = & 1, \\ x & + & y & + & 2z & = & 3, \\ x & + & y & + & 0 & = & -1. \end{array}\right.$$

(a) Put the system in RREF (Reduced Row Echelon Form).

(b) Use your answer to solve for (x, y, z).

(a): We convert the system to a matrix and then we perform Gaussian elimination:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 0 & -1 \end{array}\right) \begin{array}{l} 1 \\ 2 \\ 3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & -2 \end{array}\right) \begin{array}{l} 1 \\ 2 = 2 - 1\,1 \\ 3 = 3 - 1\,1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} 1 \\ 2 \\ 3 = 3 - 1\,2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} 1 = 1 - 1\,2 \\ 2 \\ 3 \end{array}$$

Then we convert the RREF of the matrix back into a system of linear equations:

$$\left\{\begin{array}{ccccccc} x & + & y & + & 0 & = & -1, \\ 0 & + & 0 & + & z & = & 2, \\ 0 & + & 0 & + & 0 & = & 0. \end{array}\right.$$

The pivot variables are x, z and the free variable is y. Let's say $t = y$. Then the solution is

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1-t \\ t \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \mathbf{p} + t\mathbf{v}.$$

This is a parametrized line in $\mathbb{R}^3$ containing the point $\mathbf{p} = (-1, 0, 2)$ and moving in the direction of $\mathbf{v} = (-1, 1, 0)$.

[Remark: Since we got a row of zeroes in the RREF, there must have been a linear relation among the row vectors. Can you find this relation? Hint: Compute the RREF of the transpose matrix. After doing this you will find that (row 3) = 2(row 1) − 1(row 2).]

Problem 2. [6 points] Consider a system of m linear equations in n unknowns:

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1, \\ & & & & & & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m. \end{cases}$$

Let r be the number of pivot variables and let d be the number of free (i.e., non-pivot) variables in the RREF of the system, so that $r + d = n$. Assume that the solution set is not empty.

Fill in the blanks:

(a) The solution set is a ____-dimensional plane living in ____-dimensional space.

(b) We must have $0 \leq r \leq m$ because ______________ .

(c) It follows that ____ $\leq d \leq$ ____ .

(a): The solution set is a d-dimensional plane living in n-dimensional space.

(b): We must have $0 \leq r \leq m$ because there can be at most one pivot in each row.

(c): It follows that

$$\begin{aligned} &0 \leq r \leq m \\ &0 \geq -r \geq -m \\ &n \geq n - r \geq n - m \\ &n \geq d \geq n - m. \end{aligned}$$

[Remark: In other words, the dimension of the solution is greater than or equal to the number of variables minus the number of equations. We have an equality $d = n - m$ exactly when $r = m$, i.e., when there is a pivot in every row of the RREF, i.e., when the row vectors of the corresponding matrix are independent. Example: In Problem 1 we had $m = 3$, $n = 3$, $r = 2$ and $d = 1$. The fact that $r = m - 1$ led to the fact that $d = (n - m) + 1$. In other words, since we had one fewer pivot variables than expected, the dimension of the solution was one greater than expected.]