# Math 210 — Fall 2020
## Homework 6 — Drew Armstrong

**Problem 1. An Important Formula.** Let $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$ be eigenvectors of a square matrix $A$:

$$A\mathbf{u} = \lambda \mathbf{u} \quad \text{and } A\mathbf{v} = \mu \mathbf{v}.$$

(a) Show that $A^n \mathbf{u} = \lambda^n \mathbf{u}$ for all integers $n \geq 1$.
(b) Use part (a) to show that $A^n(a\mathbf{u} + b\mathbf{v}) = s\lambda^n \mathbf{u} + b\mu^n \mathbf{v}$ for all scalars $a, b$.

**Problem 2. A Projection and a Reflection.** Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ be vectors in the plane satisfying $\mathbf{a}^T \mathbf{b} = 0$, and let $P$ be the matrix that projects onto the line $t\mathbf{a}$:

$$P = \frac{1}{\|\mathbf{a}\|^2} \mathbf{a}\mathbf{a}^T.$$

(a) Show that $\mathbf{a}$ and $\mathbf{b}$ are eigenvectors of $P$. What are the corresponding eigenvalues?
(b) Show that $\mathbf{a}$ and $\mathbf{b}$ are eigenvectors of $F = 2P - I$. What are the eigenvalues?
(c) Describe what the matrix $F$ does geometrically.

**Problem 3. Eigenvalues of a Rotation.** Consider again the rotation matrix:

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(a) Use the characteristic equation to find the complex eigenvalues of $R_\theta$.
(b) For which values of $\theta$ are the eigenvalues real? Find the eigenvectors in each case.

**Problem 4. Diagonalizing a Matrix.** Consider the following $2 \times 2$ matrix:

$$A = \frac{1}{6} \begin{pmatrix} 5 & 4 \\ 2 & -2 \end{pmatrix}.$$

(a) Solve the characteristic equation to find the eigenvalues $\lambda, \mu$.
(b) Solve the equations $(A - \lambda I)\mathbf{u} = \mathbf{0}$ and $(A - \mu I)\mathbf{v} = \mathbf{0}$ to find eigenvectors $\mathbf{u}, \mathbf{v}$.
(c) Draw a picture of the eigenspaces in the plane.

**Problem 5. Two Dynamical Systems.** Let $A$ be the same matrix from Problem 4.

(a) Express the vector $(2, 5)$ as $a\mathbf{u} + b\mathbf{v}$ where $\mathbf{u}, \mathbf{v}$ are the eigenvectors of $A$.
(b) **A Discrete Dynamical System.** Let the points $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ in $\mathbb{R}^2$ be defined by

$$\mathbf{x}_0 = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{n+1} = A\mathbf{x}_n.$$

Use part (a) and Problem 4 to find an explicit formula for $\mathbf{x}_n$. [Recall that the general solution looks like $\mathbf{x}_n = a\lambda^n \mathbf{u} + b\mu^n \mathbf{v}$.]

(c) **A Continuous Dynamical System.** Let the path $\mathbf{x}(t)$ in $\mathbb{R}^2$ be defined by[1]

$$\mathbf{x}(0) = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad \text{and} \quad \mathbf{x}'(t) = A\mathbf{x}(t).$$

Use part (a) and Problem 4 to find an explicit formula for $\mathbf{x}(t)$. [Recall that the general solution looks like $\mathbf{x}(t) = ae^{\lambda t}\mathbf{u} + be^{\mu t}\mathbf{v}$.]

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[1] If the position $\mathbf{x}(t)$ has coordinates $x(t)$ and $y(t)$ then the velocity $\mathbf{x}'(t)$ has coordinates $x'(t)$ and $y'(t)$.