Math 210 **Fall 2020**
Homework 5 **Drew Armstrong**

Problem 1. Special Matrices. For any angle θ we define the following matrices:

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad F_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \quad P_\theta = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}.$$

(a) Describe what each matrix does geometrically.
(b) Compute the determinant of each matrix.
(c) For each matrix that is invertible, compute the inverse.

Problem 2. Projections in General.[1] We call P a *projection* if $P^T = P$ and $P^2 = P$.

(a) If P is a projection, show that $Q = I - P$ is also a projection.
(b) Show that the projections P and Q from part (a) satisfy $PQ = 0$.
(c) Let A be any matrix (possibly non-square), so that A^TA is a square matrix. Assuming that $(A^TA)^{-1}$ exists, show that $P = A(A^TA)^{-1}A^T$ is a projection. [We saw in class that this matrix projects orthogonally onto the **column space** of A.]
(d) In the special case that A is invertible, show that $P = A(A^TA)^{-1}A^T = I$. What does this mean? [Hint: The column space of an invertible matrix is the whole space.]

Problem 3. Specific Projections. Consider the following matrices:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 2 \end{pmatrix}.$$

(a) Compute the 3×3 matrix $P = \mathbf{a}(\mathbf{a}^T\mathbf{a})^{-1}\mathbf{a}^T$ that projects onto the column space of $\mathbf{a}$, i.e., the matrix that projects onto the line $t(1, 1, -1)$.
(b) Compute the 3×3 matrix $Q = A(A^TA)^{-1}A^T$ that projects onto the column space of A, i.e., the matrix that projects onto the plane $s(1, 2, 3) + t(1, 1, 2)$.
(c) Check that $P + Q = I$ and $PQ = 0$. Why does this happen? [Hint: How are the line from part (a) and the plane from part (b) related to each other?]

Problem 4. Least Squares Approximation. Consider the following two lines in $\mathbb{R}^3$:

$$L_1 : (x, y, z) = (0, 0, 0) + s(1, 1, 1), \quad L_2 : (x, y, z) = (1, 0, 0) + t(-1, 1, 0).$$

(a) Write down the system of three linear equations in s, t that expresses the intersection of the two lines. [This system has no solution because the lines do **not** intersect.]
(b) Find the OLS best approximations $\hat{s}$ and $\hat{t}$ for the system in part (a).
(c) Use your answer from (b) to compute the minimum distance between the two lines.

Problem 5. Least Squares Regression. Consider four data points:

$$(x, y) = (1, 1), (2, 1), (3, 3), (4, 5).$$

(a) Find the OLS best fit line $y = mx + b$ for these points. Draw your answer.
(b) Find the OLS best fit parabola $y = ax^2 + bx + c$ for the same points. Draw your answer.

[I recommend using a computer algebra system to solve the normal equations.]

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[1] Technically, these matrices are called *orthogonal projections* because they project at right angles.